Homomesy:
Actions and Averages

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Slides for this talk are on-line at

http://jamespropp.org/uw14a.pdf

Acknowledgments

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Overview

For many invertible actions τ on a finite set S of combinatorial objects, and for many natural real-valued statistics ϕ on S, one finds that the orbit-average

$$\frac{1}{n}\sum_{i=0}^{n-1}\phi(\tau^i(x))$$

(where $\tau^n(x) = x$) is **independent** of the starting point $x \in S$.

We say that ϕ is **homomesic** (from Greek: "same middle") with respect to the combinatorial dynamical system (S, τ).

I'll give numerous examples of **homomesies** (homomesic functions), some proved and others conjectural.

Please interrupt with questions!

Introductory examples

1. Ballot sequences
2. Inversions and rotation of bit-strings
3. Promotion of rectangular Semi-Standard Young Tableaux
4. Suter's symmetries

Example 1: Ballot sequences

Let S be the set of strings composed of $n+1$ terms equal to $+1$ and n terms equal to -1, and for $s = (s_1, s_2, \ldots, s_{2n+1}) \in S$ let

$$\phi(s) = \begin{cases} 1 & \text{if } s_1 + s_2 + \cdots + s_k > 0 \text{ for all } 1 \leq k \leq 2n+1, \\ 0 & \text{otherwise.} \end{cases} .$$

That is, $\phi(s) = 1$ if s is a ballot sequence, 0 otherwise.

Let $\tau(s) = s'$ with $s' = (s_2, \ldots, s_{2n+1}, s_1)$.

A classic way to enumerate ballot sequences is to show that each τ-orbit is of size $2n+1$ and contains exactly one ballot sequence (i.e., the average of ϕ in each τ-orbit is $1/(2n+1)$).

Example 2: Inversions and rotation of bit-strings

Prop.: Let $\mathcal{O}$ be an orbit in the set of words w composed of a 0's and b 1's under the action of rotation (cyclic shift). Then

$$\frac{1}{\#\mathcal{O}} \sum_{w \in \mathcal{O}} \mathrm{inv}(w) = \frac{ab}{2}$$

where $\mathrm{inv}(w) = \#\{i, j : \ i < j, \ w_i > w_j\}$.

(E.g., $(\mathrm{inv}(0011) + \mathrm{inv}(0110) + \mathrm{inv}(1100) + \mathrm{inv}(1001))/4 =$ $(0 + 2 + 4 + 2)/4 = 2 = \frac{(2)(2)}{2}$.)

I know two simple ways to prove this: one can show pictorially that the value of the sum doesn't change when you mutate w (replacing a 01 somewhere in w by 10 or vice versa), or one can write the number of inversions in w as $\sum_{i<j} w_i(1 - w_j)$ and then perform algebraic manipulations.

Example 3: Promotion of rectangular SSYT's

First consider the case of SYT's.

Fix a rectangular Young diagram λ of size N, and for each $1 \leq i \leq N-1$, let s_i be the action on Standard Young Tableaux of shape λ that swaps the i and the $i+1$ provided that the swap results in a valid SYT, and let ∂ be the composition of the maps $s_1, s_2, \ldots, s_{N-1}$.

A small example of promotion

(taken from J. Striker and N. Williams, *Promotion and Rowmotion*, European J. Combin. 33 (2012), no. 8, 1919–1942; http://arxiv.org/abs/1108.1172):

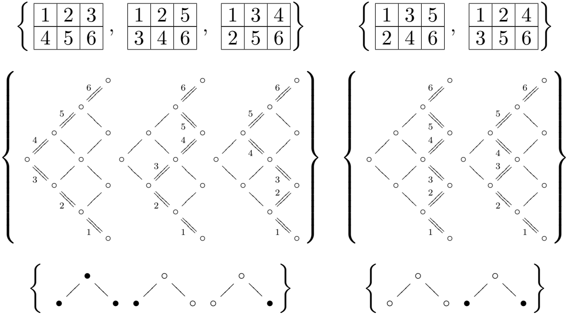


FIGURE 5. The two orbits of SYT of shape $(3, 3)$ under promotion, the san
orbits using the maximal chain interpretation, and the same two orbits usi
order ideal interpretation.

A small example of promotion: centrally symmetric sums

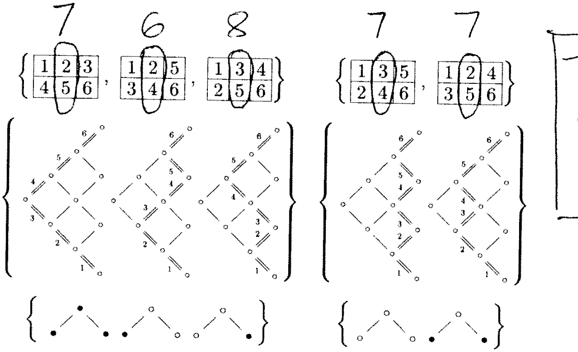


FIGURE 5. The two orbits of SYT of shape (3, 3) under promotion, the same orbits using the maximal chain interpretation, and the same two orbits using order ideal interpretation.

We apply this idea of boundary paths under ρ to noncrossing objects under rotation in and generalize it in Section 7. In Sections 7 and 8, we conjecture that there is a further gen to the type D_n positive root poset, plane partitions, the ASM poset, and the TSSCPP

Bloom, Pechenik, and Saracino's homomesy theorem

The Bender-Knuth involutions are operations on semi-standard (aka column-strict) tableaux that generalize the maps $s_1, \ldots, s_{N-1}$ discussed above:

If a tableau has a i's and b $i+1$'s, then after the ith Bender-Knuth involution is applied, the resulting tableau has b i's and a $i+1$'s.

One can define promotion on semi-standard skew tableaux with all entries $\leq N$ by successively applying the ith Bender-Knuth involution, with i going from 1 to $N-1$.

Theorem (Bloom, Pechenik, Saracino): If a tableau has rectangular shape and no entry $> N$, and $\phi(T)$ denotes the sum of the numbers in cells c and c' where cells c and c' are opposite one another (note: the case $c = c'$ is permitted when λ is odd-by-odd), then ϕ is homomesic under promotion.

Example 4: Suter's symmetries

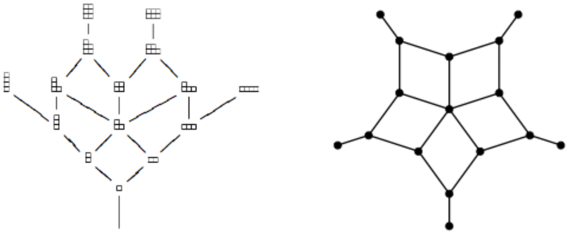
Let $\mathbb{Y}_N$ be the set of number-partitions λ whose maximal hook lengths are strictly less than N (i.e., whose Young diagrams fit inside some rectangle that fits inside the staircase shape $(N-1, N-2, ..., 2, 1)$).

Suter showed that the Hasse diagram of $\mathbb{Y}_N$ has N-fold cyclic symmetry (indeed, N-fold dihedral symmetry) by exhibiting an explicit action of order N.

Suter's action, $N = 5$

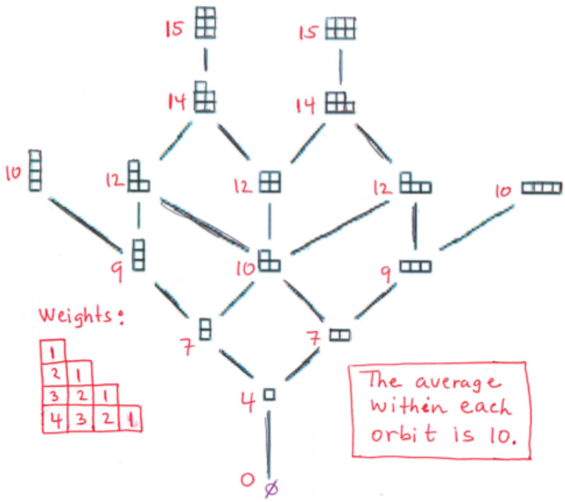
(taken from R. Suter, *Young's lattice and dihedral symmetries revisited: Möbius strips and metric geometry*; http://arxiv.org/abs/1212.4463):

Example (Hasse diagram of $\mathbb{Y}_5$ and its undirected Hasse graph)



This graph has a 5-fold (cyclic) symmetry, and its full symmetry group is a dihedral group of order 10.

Suter's action, $N = 5$: weighted sums



Suter's action: homomesies

Assign weight 1 to the cells at the diagonal boundary of the staircase shape, weight 2 to their neighbors, ..., and weight $N-1$ to the cell at the lower left, and for $\lambda \in \mathbb{Y}_N$ let $\phi(\lambda)$ be the sum of the weights of all the cells in the Young diagram of λ.

Prop. (Einstein, P.): ϕ is homomesic under Suter's map with average value $(n^3 - n)/12$.

More refined result: If $i + j = N$ (note: $i = j$ is permitted), and $\phi_{i,j}(\lambda)$ is the sum of the weights of all the cells in λ with weight i plus the sum of the weights of all the cells in λ with weight j, then $\phi_{i,j}$ is homomesic under Suter's map with average ij in all orbits.

The main part of the talk

The Panyushev complement

Antichains in $[a] \times [b]$

Order ideals in $[a] \times [b]$

An invertible operation on antichains

Let $\mathcal{A}(P)$ be the set of antichains of a finite poset P.

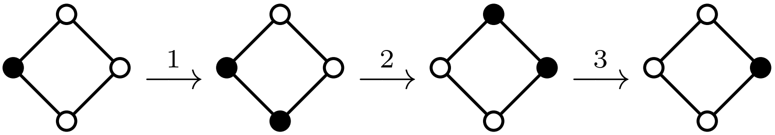
Given $A \in \mathcal{A}(P)$, let $\tau(A)$ be the set of minimal elements of the complement of the downward-saturation of A.
τ is invertible since it is a composition of three invertible operations:

$$\text{antichains} \longleftrightarrow \text{downsets} \longleftrightarrow \text{upsets} \longleftrightarrow \text{antichains}$$

This map and its inverse have been considered with varying degrees of generality, by many people more or less independently (using a variety of nomenclatures and notations): Duchet, Brouwer and Schrijver, Cameron and Fon Der Flaass, Fukuda, Panyushev, Rush and Shi, and Striker and Williams.

An example

1. Saturate downward
2. Complement
3. Take minimal element(s)



(For a bigger example, see the example of rowmotion on slide 4 of `http://www.math.uconn.edu/~troby/combErg2012kizugawa.pdf`.)

Panyushev's conjecture

Let Δ be a reduced irreducible root system in $\mathbf{R}^n$.
Choose a system of positive roots and make it a poset of rank n by decreeing that y covers x iff $y - x$ is a simple root.
Conjecture (Conjecture 2.1(iii) in D.I. Panyushev, *On orbits of antichains of positive roots*, European J. Combin. 30 (2009), 586-594): Let $\mathcal{O}$ be an arbitrary τ-orbit. Then

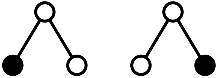
$$\frac{1}{\#\mathcal{O}} \sum_{A \in \mathcal{O}} \#(A) = \frac{n}{2}.$$

(Two other assertions of this kind, Panyushev's Conjectures 2.3(iii) and 2.4(ii), appear to remain open.)

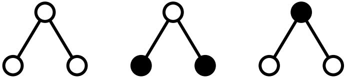
Panyushev's Conjecture 2.1(iii) (along with much else) was proved by Armstrong, Stump, and Thomas in their article *A uniform bijection between nonnesting and noncrossing partitions*, http://arxiv.org/abs/1101.1277.

Panyushev's conjecture: The A_n case, $n = 2$

Here we have just an orbit of size 2 and an orbit of size 3:



$1 \qquad 1$



$0 \qquad 2 \qquad 1$

Within each orbit, the average antichain has cardinality $n/2 = 1$.

Antichains in $[a] \times [b]$: cardinality is homomesic

A simpler-to-prove phenomenon of this kind concerns the poset $[a] \times [b]$ (where $[k]$ denotes the linear ordering of $\{1, 2, \ldots, k\}$):

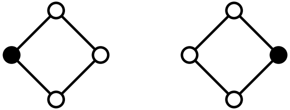
Theorem (P., Roby): Let $\mathcal{O}$ be an arbitrary τ-orbit in $\mathcal{A}([a] \times [b])$. Then

$$\frac{1}{\#\mathcal{O}} \sum_{A \in \mathcal{O}} \#(A) = \frac{ab}{a+b}.$$

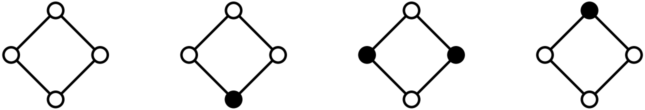
This is an easy consequence of unpublished work of Hugh Thomas building on earlier work of Richard Stanley: see the last paragraph of section 2 of R. Stanley, *Promotion and evacuation*, http://www.combinatorics.org/ojs/index.php/eljc/article/view/v16i2r9 .

Antichains in $[a] \times [b]$: the case $a = b = 2$

Here we have an orbit of size 2 and an orbit of size 4:



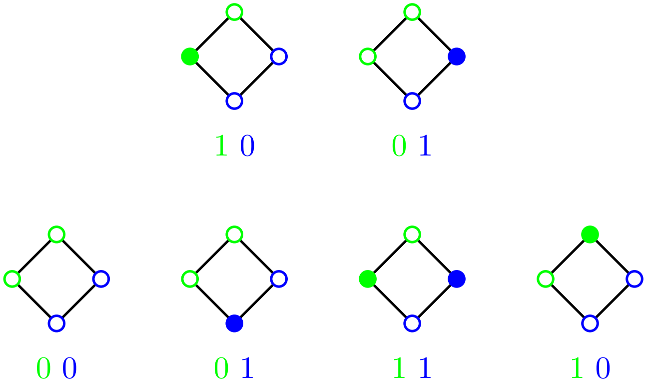
$1 \qquad 1$



$0 \qquad 1 \qquad 2 \qquad 1$

Within each orbit, the average antichain has cardinality $ab/(a+b) = 1$.

Antichains in $[a] \times [b]$: fiber-cardinality is homomesic



Within each orbit, the average antichain has
$1/2$ a green element and $1/2$ a blue element.

Antichains in $[a] \times [b]$: fiber-cardinality is homomesic

For $(i,j) \in [a] \times [b]$, and A an antichain in $[a] \times [b]$, let $1_{i,j}(A)$ be 1 or 0 according to whether or not A contains (i,j).

Also, let $f_i(A) = \sum_{j \in [b]} 1_{i,j}(A) \in \{0,1\}$ (the cardinality of the intersection of A with the fiber $\{(i,1),(i,2),\ldots,(i,b)\}$ in $[a] \times [b]$), so that $\#(A) = \sum_i f_i(A)$.
Likewise let $g_j(A) = \sum_{i \in [a]} 1_{i,j}(A)$, so that $\#(A) = \sum_j g_j(A)$.

Theorem (P., Roby): For all i, j,

$$\frac{1}{\#\mathcal{O}} \sum_{A \in \mathcal{O}} f_i(A) = \frac{b}{a+b} \qquad \text{and} \qquad \frac{1}{\#\mathcal{O}} \sum_{A \in \mathcal{O}} g_j(A) = \frac{a}{a+b}.$$

The indicator functions f_i and g_j are homomesic under τ, even though the indicator functions $1_{i,j}$ aren't.

Antichains in $[a] \times [b]$: centrally symmetric homomesies

Theorem (P., Roby): In any orbit, the number of A that contain (i, j) equals the number of A that contain the opposite element $(i', j') = (a + 1 - i, b + 1 - j)$.

That is, the function $1_{i,j} - 1_{i',j'}$ is homomesic under τ, with average value 0 in each orbit.

Linearity

Useful triviality: every linear combination of homomesies is itself homomesic.

E.g., consider the adjusted major index statistic defined by $\mathrm{amaj}(A) = \sum_{(i,j) \in A} (i - j)$.

P. and Roby proved that amaj is homomesic under τ
by writing it as a linear combination of the functions $1_{i,j} - 1_{i',j'}$.
Haddadan gave a simpler proof,
writing amaj as a linear combination of the functions f_i and g_j.

~~**Question**~~: Are there other homomesic combinations of the indicator functions $1_{i,j}$ (with $(i,j) \in [a] \times [b]$),
linearly independent of the functions f_i, g_j, and $1_{i,j} - 1_{i',j'}$?

Theorem (Einstein): No.

From antichains to order ideals

Given a poset P and an antichain A in P, let $\mathcal{I}(A)$ be the order ideal $I = \{y \in P \ : y \leq x \text{ for some } x \in A\}$ associated with A, so that for any order ideal I in P, $\mathcal{I}^{-1}(I)$ is the antichain of maximal elements of I.

As usual, we let $J(P)$ denote the set of (order) ideals of P.

We define $\overline{\tau} : J(P) \to J(P)$ by $\overline{\tau}(I) = \mathcal{I}(\tau(\mathcal{I}^{-1}(I)))$. That is, $\overline{\tau}(I)$ is the downward saturation of the set of minimal elements of the complement of I.

For $(i,j) \in P$ and $I \in J(P)$, let $\overline{1}_{i,j}(I)$ be 1 or 0 according to whether or not I contains (i,j).

One action, two vector spaces

$\overline{\tau}$ is "the same" τ in the sense that the standard bijection from $\mathcal{A}(P)$ to $J(P)$ (downward saturation) makes the following diagram commute:

$$\begin{array}{ccc} \mathcal{A}(P) & \xrightarrow{\tau} & \mathcal{A}(P) \\ \downarrow & & \downarrow \\ J(P) & \xrightarrow{\overline{\tau}} & J(P) \end{array}$$

However, the bijection from $\mathcal{A}(P)$ to $J(P)$ does **not** carry the vector space generated by the functions $1_{i,j}$ to the vector space generated by the functions $\overline{1}_{i,j}$ in a linear way.

So the homomesy situation for $\overline{\tau} : J(P) \to J(P)$ could be (and, as we'll see, is) different from the homomesy situation for $\tau : \mathcal{A}(\mathcal{P}) \to \mathcal{A}(\mathcal{P})$.

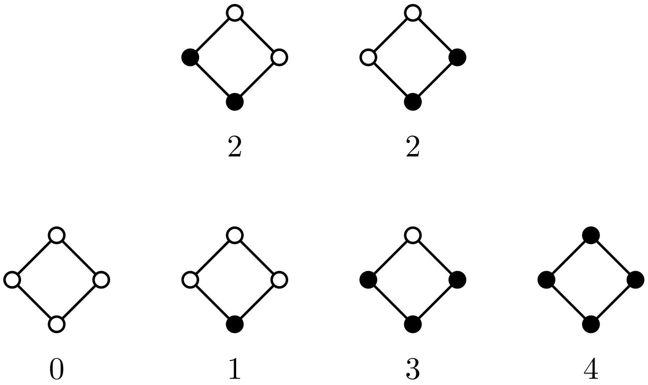
Ideals in $[a] \times [b]$: cardinality is homomesic

Theorem (P., Roby): Let $\mathcal{O}$ be an arbitrary $\overline{\tau}$-orbit in $J([a] \times [b])$. Then

$$\frac{1}{\#\mathcal{O}} \sum_{I \in \mathcal{O}} \#(I) = \frac{ab}{2}.$$

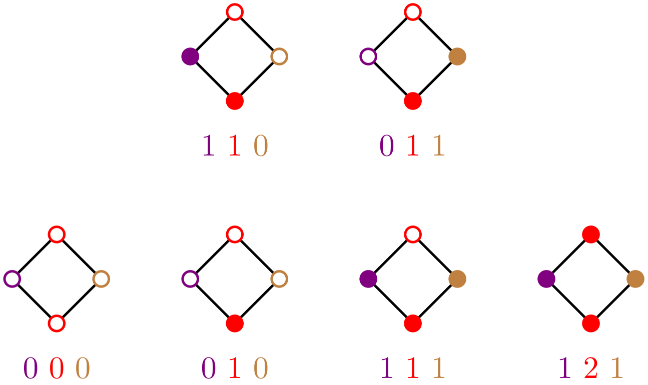
Ideals in $[a] \times [b]$: the case $a = b = 2$

Again we have an orbit of size 2 and an orbit of size 4:



Within each orbit, the average order ideal has cardinality $ab/2 = 2$.

Ideals in $[a] \times [b]$: file-cardinality is homomesic



Within each orbit, the average order ideal has
$1/2$ a violet element, 1 red element, and $1/2$ a brown element.

Ideals in $[a] \times [b]$: file-cardinality is homomesic

For $1 - b \leq k \leq a - 1$, define the kth **file** of $[a] \times [b]$ as

$$\{(i,j) : 1 \leq i \leq a,\ 1 \leq j \leq b,\ i - j = k\}.$$

For $1 - b \leq k \leq a - 1$, let $h_k(I)$ be the number of elements of I in the kth file of $[a] \times [b]$, so that $\#(I) = \sum_k h_k(I)$.

Theorem (P., Roby): For every $\overline{\tau}$-orbit $\mathcal{O}$ in $J([a] \times [b])$,

$$\frac{1}{\#\mathcal{O}} \sum_{I \in \mathcal{O}} h_k(I) = \begin{cases} \frac{(a-k)b}{a+b} & \text{if } k \geq 0 \\ \frac{a(b+k)}{a+b} & \text{if } k \leq 0. \end{cases}$$

Ideals in $[a] \times [b]$: centrally symmetric homomesies

Recall that for $(i, j) \in [a] \times [b]$, and I an ideal in $[a] \times [b]$, $\overline{1}_{i,j}(I)$ is 1 or 0 according to whether or not I contains (i, j).

Write $(i', j') = (a + 1 - i, b + 1 - j)$, the point opposite (i, j) in the poset.

Theorem (P., Roby): $\overline{1}_{i,j} + \overline{1}_{i',j'}$ is homomesic under $\overline{\tau}$.

~~**Question**~~: In addition to the functions h_k and $\overline{1}_{i,j} + \overline{1}_{i',j'}$, are there other homomesic functions in the span of the functions $\overline{1}_{i,j}$?

Theorem (Einstein): No.

The two vector spaces, compared

In the space associated with antichains:

  fiber-cardinalities and

  centrally symmetric **differences**

are homomesic.

In the space associated with order ideals:

  file-cardinalities and

  centrally symmetric **sums**

are homomesic.

Extra topics

Toggling

Other actions

Other posets

Continuous piecewise-linear maps

Connections between Striker-and-Williams promotion and Schützenberger promotion

Birational maps

Non-periodic actions

Toggling

In their 1995 article *Orbits of antichains revisited*, European J. Combin. 16 (1995), 545–554, Cameron and Fon-der-Flaass give an alternative description of $\overline{\tau}$.

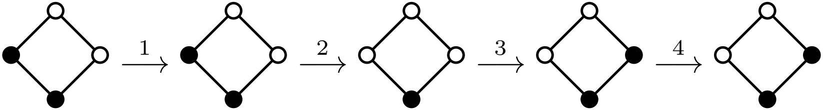
Given $I \in J(P)$ and $x \in P$, let $\tau_x(I) = I \triangle \{x\}$ provided that $I \triangle \{x\}$ is an order ideal of P; otherwise, let $\tau_x(I) = I$.

We call the involution τ_x "toggling at x".

The involutions τ_x and τ_y commute unless x covers y or y covers x.

An example

1. Toggle the top element
2. Toggle the left element
3. Toggle the right element
4. Toggle the bottom element



Toggling from top to bottom

Theorem (Cameron and Fon-der-Flaass): Let $x_1, x_2, \ldots, x_n$ be any order-preserving enumeration of the elements of the poset P. Then the action on $J(P)$ given by the composition $\tau_{x_1} \circ \tau_{x_2} \circ \cdots \circ \tau_{x_n}$ coincides with the action of $\overline{\tau}$.

In the particular case $P = [a] \times [b]$, we can enumerate P rank-by-rank; that is, we can list the (i,j)'s in order of increasing $i+j$.

Note that all the involutions coming from a given rank of P commute with one another, since no two of them are in a covering relation.

Striker and Williams refer to $\overline{\tau}$ (and τ) as **rowmotion**, since for them, "row" means "rank".

Toggling from side to side

Recall that a file in $P = [a] \times [b]$ is the set of all $(i, j) \in P$ with $i - j$ equal to some fixed value k.

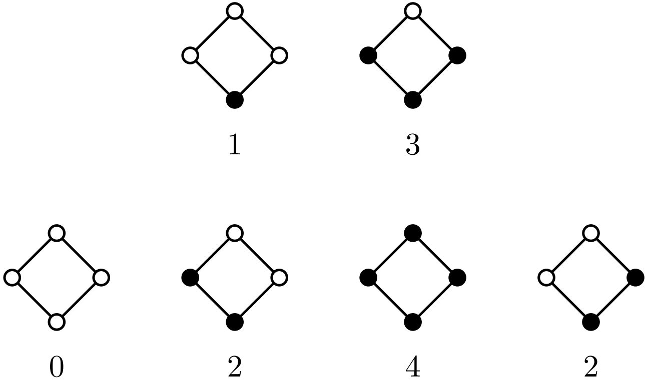
Note that all the involutions coming from a given file commute with one another, since no two of them are in a covering relation.

It follows that for any enumeration $x_1, x_2, \ldots, x_n$ of the elements of the poset $[a] \times [b]$ arranged in order of increasing $i - j$, the action on $J(P)$ given by $\tau_{x_1} \circ \tau_{x_2} \circ \cdots \circ \tau_{x_n}$ doesn't depend on which enumeration was used.

Striker and Williams call this well-defined composition **promotion**, and denote it by ∂, since it is closely related to Schützenberger's notion of promotion on linear extensions of posets.

Promoting ideals in $[a] \times [b]$: the case $a = b = 2$

Again we have an orbit of size 2 and an orbit of size 4:



$J([a] \times [b])$: cardinality is homomesic under promotion

Claim (P., Roby): Let $\mathcal{O}$ be an arbitrary orbit in $J([a] \times [b])$ under the action of promotion ∂. Then

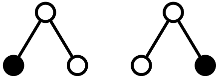
$$\frac{1}{\#\mathcal{O}} \sum_{I \in \mathcal{O}} \#(I) = \frac{ab}{2}.$$

The result about cyclic rotation of binary words discussed earlier ("Example 2") turns out to be a special case of this.

Root posets of type A: antichains

Recall that, by the Armstrong-Stump-Thomas theorem, the cardinality of antichains is homomesic under the action of rowmotion, where the poset P is a root poset of type A_n.

E.g., for $n = 2$:



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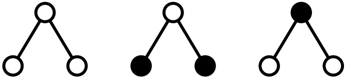


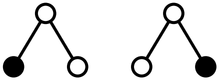
0 2 1

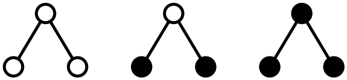
Antichain-cardinality is homomesic: in each orbit, its average is 1.

Root posets of type A: order ideals

What if instead of antichains we take order ideals?

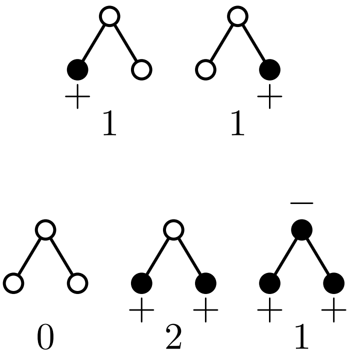
E.g., $n = 2$:





What is homomesic here?

Root posets of type A: rank-signed cardinality



Root posets of type A: rank-signed cardinality is homomesic

Theorem (Haddadan): Let P be the root poset of type A_n. If we assign an element $x \in P$ weight $\mathrm{wt}(x) = (-1)^{\mathrm{rank}(x)}$, and assign a order ideal $I \in J(P)$ weight $\phi(I) = \sum_{x \in I} \mathrm{wt}(x)$, then ϕ is homomesic under rowmotion and promotion, with average $n/2$.

The order polytope of a poset

Let P be a poset, extended to $\hat{P}$ by adjoining an extra minimal element $\hat{0}$ and an extra maximal element $\hat{1}$.

The order polytope $\mathcal{O}(P)$ (introduced by R. Stanley) is the set of functions $f : \hat{P} \to [0,1]$ with $f(\hat{0}) = 0$, $f(\hat{1}) = 1$, and $f(x) \leq f(y)$ whenever $x \leq y$ in $\hat{P}$.

Flipping-maps in the order polytope

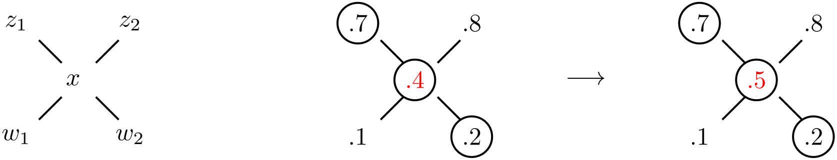
For each $x \in P$, define the flip-map $\sigma_x : \mathcal{O}(P) \to \mathcal{O}(P)$ sending f to the unique f' satisfying

$$f'(y) = \begin{cases} f(y) & \text{if } y \neq x, \\ \min_{z \cdot > x} f(z) + \max_{w < \cdot x} f(w) - f(x) & \text{if } y = x, \end{cases}$$

where $z \cdot > x$ means z covers x and $w < \cdot x$ means x covers w (here we allow w to be $\hat{0}$ and z to be $\hat{1}$).

Note that the interval $[\min_{z \cdot > x} f(z), \max_{w < \cdot x} f(w)]$ is precisely the set of values that $f'(x)$ could have so as to satisfy the order-preserving condition, if $f'(y) = f(y)$ for all $y \neq x$; the map that sends $f(x)$ to $\min_{z \cdot > x} f(z) + \max_{w < \cdot x} f(w) - f(x)$ is just the affine involution that swaps the endpoints.

Example



$$\min_{z \cdot > x} f(z) + \max_{w < \cdot x} f(w) = .7 + .2 = .9$$

$$f(x) + f'(x) = .4 + .5 = .9$$

Flipping and toggling

If we associate each order-ideal I with the indicator function of $P \setminus I$ (that is, the function that takes the value 0 on I and the value 1 everywhere else), then toggling I at x is tantamount to flipping f at x.

That is, we can identify $J(P)$ with the vertices of the polytope $\mathcal{O}(P)$ in such a way that toggling can be seen to be a special case of flipping.

This may be clearer if you think of $J(P)$ as being in bijection with the set of monotone 0,1-valued functions on P.

Flipping

Flipping (at least in special cases) is not new, though it is not well-studied; the most worked-out example I've seen is Berenstein and Kirillov's article *Groups generated by involutions, Gelfand-Tsetlin patterns and combinatorics of Young tableaux* (St. Petersburg Math. J. 7 (1996), 77–127); see http://pages.uoregon.edu/arkadiy/bk1.pdf.

By writing semi-standard Young tableaux as Gelfand-Testlin patterns, we can view SSYT's of fixed shape and with largest entry N as lattice points in a polytope, and thereby view Bender-Knuth involutions (and Schützenberger promotion) as flipping involutions (and compositions thereof) restricted to the lattice points in the polytope. See `http://jamespropp.org/fpsac14.pdf`.

Composing flips

Just as we can apply toggle-maps from top to bottom, we can apply flip-maps from top to bottom:

$$
\begin{array}{ccc}
 & .8 & \\
.4 & & .3 \\
 & .1 &
\end{array}
\qquad
\begin{array}{ccc}
 & .6 & \\
.4 & & .3 \\
 & .1 &
\end{array}
\qquad
\begin{array}{ccc}
 & .6 & \\
.3 & & .3 \\
 & .1 &
\end{array}
$$

$$
\begin{array}{ccc}
 & .6 & \\
.3 & & .4 \\
 & .1 &
\end{array}
\qquad
\begin{array}{ccc}
 & .6 & \\
.3 & & .4 \\
 & .2 &
\end{array}
$$

(Here, as we read across the first row of arrays and then the second, we successively flip values at the North, West, East, and South.)

Homomsey

All of the aforementioned results on homomesy for rowmotion and promotion on $J([a] \times [b])$ lift to corresponding results in the order polytope, where instead of composing toggle-maps to obtain rowmotion and promotion we compose the corresponding flip-maps to obtain c.p.l. (continuous piecewise-linear) maps from $\mathcal{O}([a] \times [b])$ to itself.

It turns out that the hardest part is showing that rowmotion and promotion on $\mathcal{O}([a] \times [b])$, defined as above, are maps of order $a + b$. This was done by Grinberg and Roby. It would be good to have a simpler proof.

Example

An orbit of c.p.l. rowmotion (flipping values from top to bottom):

$$
\begin{array}{ccc} & .7 & \\ .2 & & .4 \\ & .1 & \end{array}
\qquad
\begin{array}{ccc} & .7 & \\ .6 & & .4 \\ & .3 & \end{array}
\qquad
\begin{array}{ccc} & .9 & \\ .6 & & .8 \\ & .3 & \end{array}
\qquad
\begin{array}{ccc} & .9 & \\ .6 & & .4 \\ & .1 & \end{array}
$$

The average is

$$
\begin{array}{ccc} & .8 & \\ .5 & & .5 \\ & .2 & \end{array}
$$

Within this orbit (and every other), the violet averages sum to $1/2$, the red averages sum to 1, and the brown averages sum to $1/2$.

Continuous piecewise-linear maps

Not only does the polytope perspective allow us to see toggling as the restriction of a c.p.l. map, but it also lets us see the bijection from $J(P)$ to $\mathcal{A}(P)$ as the restriction of the c.p.l. map $f \mapsto g$ where $g(x) = \min_{x \cdot > y}(f(x) - f(y))$ (this is Stanley's bijection between the order polytope and the chain polytope).

This allows us to lift rowmotion on $\mathcal{A}(P)$ to a polytope action, and experiments suggest that all of the results on homomesy for rowmotion on $\mathcal{A}([a] \times [b])$ lift to corresponding results in a polytope.

Example (continued):

$$
\begin{array}{ccc}
 & .3 & \\
.1 & & .3 \\
 & .1 &
\end{array}
\qquad
\begin{array}{ccc}
 & .1 & \\
.3 & & .1 \\
 & .3 &
\end{array}
\qquad
\begin{array}{ccc}
 & .1 & \\
.3 & & .5 \\
 & .3 &
\end{array}
\qquad
\begin{array}{ccc}
 & .3 & \\
.5 & & .3 \\
 & .1 &
\end{array}
$$

The average is

$$
\begin{array}{ccc}
 & \color{green}{.2} & \\
\color{green}{.3} & & \color{blue}{.3} \\
 & \color{blue}{.2} &
\end{array}
$$

And indeed the green numbers sum to $1/2$, as do the blue numbers.

The birational story

Just as combinatorial rowmotion and promotion on order ideals can be profitably seen as a special case of piecewise-linear rowmotion and promotion (restricted to the vertices of the order polytope), the piecewise-linear operations can be profitably seen as tropicalizations of birational analogues of rowmotion and promotion.

A rational map $f : \mathbf{C}^n \to \mathbf{C}^n$ is birational if there's a rational map $g : \mathbf{C}^n \to \mathbf{C}^n$ such that $f \circ g$ and $g \circ f$ are the identity map on a dense open subset of $\mathbf{C}^n$.

E.g., $f((x,y)) = (x, (x^3+1)/y)$ is birational and is its own inverse on a dense open set because $(x^3+1)/((x^3+1)/y) = y$ as long as $y \neq 0$.

Technically speaking f and g aren't functions on all of $\mathbf{C}^n$, but we'll gloss over that nicety.

Birational flipping

De-tropicalization dictionary: Replace $+$, $-$, max and min by $\times$, $/$, $+$ and $\|$, where $\|$ is "parallel addition":

$$a \parallel b = ab/(a+b)$$

For each $x \in P$, define the birational flip-map $\hat{\sigma}_x : \mathbf{C}^{|P|} \to \mathbf{C}^{|P|}$ sending f to the unique f' satisfying

$$f'(y) = \begin{cases} f(y) & \text{if } y \neq x, \\ \sum^{+}_{\{f(z) : z \cdot > x\}} + \sum^{\|}_{\{f(w) : w < \cdot x\}} - f(x) & \text{if } y = x, \end{cases}$$

where $\sum^{+}_{\{r_1, r_2, \dots\}}$ denotes $r_1 + r_2 + \dots$ and $\sum^{\|}_{\{r_1, r_2, \dots\}}$ denotes $r_1 \parallel r_2 \parallel \dots$.

Birational rowmotion and promotion

Define birational rowmotion as compositions of birational flip-maps (top to bottom and left to right, respectively).

Grinberg and Roby showed that the birational versions of rowmotion and promotion have order $a+b$; the corresponding assertions about c.p.l. rowmotion and promotion follow immediately.

Einstein and Propp showed that all the homomesies of the combinatorial versions of rowmotion and promotion lift to homomesies of the birational versions; e.g., for $P = [2] \times [2]$, birational rowmotion as a map from (a dense open subset of) $\mathbf{C}^4$ to itself has the function $(w, x, y, z) \mapsto \log |wxyz|$ as a homomesy.

Infinite orbits

Let $P = [2] \times [2]$. One can show (by brute force if necesary) that the c.p.l. maps

$$\sigma_{(1,1)} \circ \sigma_{(1,2)} \circ \sigma_{(2,1)} \circ \sigma_{(2,2)}$$

("lifted rowmotion") and

$$\sigma_{(2,1)} \circ \sigma_{(1,1)} \circ \sigma_{(2,2)} \circ \sigma_{(1,2)}$$

("lifted promotion") are each of order 4.

Theorem (Einstein): The c.p.l. map

$$\omega = \sigma_{(1,1)} \circ \sigma_{(1,2)} \circ \sigma_{(2,2)} \circ \sigma_{(2,1)}$$

(flipping values in clockwise order, as opposed to going by rows or columns of P) is of infinite order.

Homomesies for infinite orbits?

Conjecture: The homomesy results for $J([2] \times [2])$ apply here too. (Note: now the relevant notion of average is the ergodic average

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n} \phi(\omega^i(x))$$

since the space no longer consists of finite orbits.)

Experimental evidence strongly supports this conjecture.
It cannot be derived from the ergodic theorem, since the map in question is not ergodic.

Taking $P = [2] \times [2]$ is just a way of getting our foot in the door; I expect $[a] \times [b]$ to exhibit similar behavior.

And...

I've found lots of examples of conjectural homomesies in all branches of combinatorics, starting at the level of the twelve-fold way and progressing through spanning trees, parking functions, abelian sandpiles (aka chip-firing), rotor-routing, etc.

There'll be an AIM workshop on this topic in Spring 2015.

For more information

See:

http://jamespropp.org/ucbcomb12.pdf
http://jamespropp.org/mathfest12a.pdf
http://www.math.uconn.edu/~troby/combErg2012kizugawa.pdf
http://jamespropp.org/mitcomb13a.pdf
http://jamespropp.org/propp-roby.pdf
http://jamespropp.org/fpsac14.pdf
http://arxiv.org/abs/1402.6178
http://arxiv.org/abs/1308.0546

Slides for this talk are on-line at

http://jamespropp.org/uw14a.pdf