A Test of the Universality Conjecture

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May, 1995

Random Domino Tilings

Consider the grid of unit squares in the coordinate plane whose vertices have integer coordinates. The squares within the lines $y = \pm n$ and $x = \pm n$ form a large square region with $4n^2$ unit squares. There are many ways to partition this region into subregions of two adjacent squares, called *dominos.* A partition into dominos is called a tiling. What do we expect to see if, from all these different domino tilings of the big square, we choose one uniformly at random? Burton and Pemantle [1] proved that in the limit as the square grows large, then the probability of seeing any particular configuration of dominos is the same at any two points in the square, provided these two points are well away from the boundary. Of course, many regions other than squares exhibit domino tilings, and for these too, we can consider random tilings. In general, the probabilities of finding certain domino configurations at a point are called "local statistics". The parti! cular local statistics at points in square regions are the same throughout the square, away from the boundary. These are known as Burton-Pemantle statistics.

Of special interest is the region known as the Aztec diamond, defined in [3]. An order n Aztec diamond is that region of lattice squares surrounded by the four lines $y = -x+(n+1), y = x-(n+1), y = x+(n+1), y = -x-(n+1)$ in the coordinate plane. When n is large, the Aztec diamond exhibits strikingly non-homogeneous behavior. Its boundary takes the shape of a square. The region between the boundary of the diamond and its inscribed circle is absolutely regular–in each of the four connected areas at the corners, all the dominoes are aligned in the same direction. Within the circle, one sees a smooth transition between the four corner behaviors. See Figure 1. We call an Aztec diamond of order n an *idealized* Aztec diamond in the limit as n grows large.

The Universality Conjecture

The Universality Conjecture, proposed by Propp in 1992, asserts the following. For any region tilable by dominos, the local statistics at a point away from the boundary will, as the size of the region increases without bound, match the local statistics at some point of the idealized Aztec diamond. For example, if it is proved that Burton-Pemantle statistics arise at the center of the idealized Aztec diamond, that will indicate that square regions satisfy the Universality Conjecture. An exception is the class of regions which have "zero entropy," as described in Section 8 of [2].

An important assumption here is that the properties of random domino tilings of a region do not change much due to a small increase in the size of the region, as long as the boundary pattern remains the same, so that the height function, as defined in [2], is only slightly changed. We believe this from experimental experience, but it has been fully proved only for squares and Aztec diamonds, along with other families of zero-entropy regions or regions with homogeneous local statistics.

The local statistics investigated in this experiment are the domino placement probabilities at a given point of a tilable region R. We distinguish four different domino orientations corresponding to the four dominos that can cover a given square in the interior of R. To be precise, color the vertices of the squares in the region alternately black and white, so that adjacent vertices have different colors. Orient every edge in the lattice with an arrow that points from the white vertex to the black vertex. The orientation of any domino is defined to be the orientation of the edge between its two squares. We call the different orientations north-going $(+y)$, east-going $(+x)$, south-going $(-y)$, and west-going $(-x)$.

Define $p_n(x,y)$ to be the probability that in a random tiling of R, the unit square centered at $(x+0.5, y+0.5)$ is in a north-going domino. Define p_e, p_s, and p_w similarly. Since every tiling has *some* domino covering the square at (x,y), and the orientations are mutually exclusive, we have the condition that $p_n + p_e + p_s + p_w = 1$.

Cohn, Elkies, and Propp [2] discovered an asymptotic formula for p_n in an Aztec diamond that is exact in the limit of the idealized Aztec diamond. The symmetry of the Aztec diamond can be exploited to produce corresponding formulas for the other three placement probabilities. The result is called the Arctangent Law, and the formula is the following:

$$p_n(x,y) = 0.5 + \frac{1}{\pi} \arctan \frac{2y-1}{\sqrt{1-2x^2-2y^2}}$$

This is the result for an idealized Aztec diamond centered at the origin, with vertices at $(0, \pm 1)$ and $(\pm 1, 0)$ and oriented so that its top two squares form a north-going domino.

There is a neat way to characterize quadruples (p_n, p_e, p_s, p_w) that belong to points in the idealized Aztec diamond. Define the *inferred bias* of such a probability quadruple as

$$B = \frac{\sin(\pi p_e)\sin(\pi p_w)}{\sin(\pi p_e)\sin(\pi p_w) + \sin(\pi p_n)\sin(\pi p_s)}.$$

The quadruples belonging to points of an idealized Aztec diamond are exactly those quadruples whose inferred bias is $B = 0.5$.

If the Universality Conjecture applies to the region R, then for any point (x, y) belonging to R, the probability quadruple $(p_n, p_e, p_s, p_w)(x, y)$ must match the quadruple of some point (X, Y) in the idealized Aztec diamond. In particular, the inferred biases of the quadruples at all such points should be 0.5. To test the validity of this last statement, we perform the following experiment.

Procedure

The region I chose to study is known as a Mayan square. Take a square of side length n. Proceeding clockwise, delete every fourth square (beginning with the third) on each edge. The region obtained in this manner is the Mayan square of order n. See Figure 2.

Random tilings for the Mayan square of order 102 are obtained by an algorithm developed by Propp and Wilson for generating random domino tilings. This method scans a current tiling for 2×2 blocks tiled by two dominos. At each location, the dominos within the block will be rotated by 90 degrees with probability 0.5 to obtain the next tiling. The process is started on the two extreme tilings at some time in the past, and at time $t = 0$ we check to see if the tilings have coupled. If not, the process is repeated from an earlier time in the past. Details are given in [4]. This algorithm is implemented by a sequence of programs created in the UROP. The output of these programs was tested for randomness in two ways. We generated many tilings of small regions and explicitly counted the number of times each tiling appeared. Also, we computed domino placement probabilities for squares and Aztec diamonds and compared the results to the exact values. One final pro! gram measures the domino orientations in each of the random tilings to create a probability quadruple for the four domino orientations.

The Mayan square is coordinatized so that its boundaries are at $x = -51, y = -51, x = +51, y = +51$. From information from 170 random tilings of the order 102 Mayan square, we measure the inferred bias at the thirty-six specific points(x, y) given by $x, y \in \{\pm 6, \pm 18, \pm 30\}$, which lie away from the boundary.

Even if Universality holds for the Mayan square, deviations from 0.5 (the target bias in an idealized Mayan square) arise due to the finiteness of the region and random sampling error. If the errors we might expect from finiteness and random sampling suffice to account for the observed deviations, then the experiment provides evidence for the Universality Conjecture.

The Aztec diamond is known to satisfy Universality. So, if we perform the same experiment on an Aztec diamond of order 72, which has approximately the same area, we again expect to find differences between 0.5 and the observed inferred bias, and again, these are due to finiteness of the diamond and random sampling error. The error due to finiteness should be about the same as for the

Mayan square, since we have chosen the same dimensions for these regions.

The Aztec diamond is coordinatized so that its center lies at the origin, and the points are chosen at locations corresponding to the thirty-six locations in the Mayan square. That is, if the Mayan square is rotated into the Aztec diamond, the points of interest will rotate into the points we study in the Aztec diamond. The coordinates given in the table of bias errors are the coordinates of the lower left vertex of the lattice square.

For Aztec diamonds, there is a faster algorithm [3] for producing random tilings. It proceeds inductively, creating a tiling of an order n Aztec diamond from a random tiling of an order $n-1$ Aztec diamond. The program developed in the UROP to tile Aztec diamonds randomly is capable of making a number of different tilings and producing the probability quadruple for domino orientations at any location in the diamond.

For each of the seventy-two sets of quadruples from the Mayan square and the Aztec diamond, inferred bias was measured and the absolute difference from 0.5 recorded.

These two sets of errors were analyzed for mean, median, and standard deviation. The Mayan square statistics were then compared with the Aztec diamond statistics, to see whether they were statistically similar.

Results

Here are the results obtained in the experiment.

Mayan Square.

point bias error

(-30, 30) .0113
(-18, 30) .0275
(-6, 30) .0152
(6, 30) .1101
(18, 30) .0191
(30, 30) .0484
(-30, 18) .2027
(-18, 18) .0699
(-6, 18) .0257
(6, 18) .0048
(18, 18) .0350
(30, 18) .0850
(-30, 6) .0134
(-18, 6) .0237
(-6, 6) .0829
(6, 6) .1743
(18, 6) .0991
(30, 6) .0025
(-30, -6) .0364
(-18, -6) .0212
(-6, -6) .0631
(6, -6) .0761
(18, -6) .0016
(30, -6) .0495
(-30, -18) .1242
(-18, -18) .0377
(-6, -18) .0302
(6, -18) .0145
(18, -18) .1240
(30, -18) .0014
(-30, -30) .0239
(-18, -30) .0505
(-6, -30) .1110
(6, -30) .0697
(18, -30) .0041
(30, -30) .0833

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Aztec Diamond.

point bias error

(-44, 0) .0112
(-34, 8) .0984
(-26, 17) .0710
(-18, 25) .0806
(-10, 33) .0710
(0, 43) .1218
(-35, -9) .0411
(-26, 0) .0616
(-18, 8) .1147
(-10, 17) .0238
(0, 25) .0157
(8, 33) .0501
(-26, -18) .2007
(-18, -9) .0288
(-9, 0) .0175
(0, 8) .0486
(9, 17) .0629
(17, 25) .0277
(-18, -26) .1435
(-10, -18) .0145
(0, -9) .0322
(8, 0) .0684
(17, 8) .0145
(25, 17) .1477
(-9, -34) .1467
(0, -26) .0266
(9, -18) .0461
(17, -9) .0530
(25, 0) .0095
(33, 8) .0671
(0, -44) .0693
(8, -34) .0414
(17, -26) .1164
(25, -18) .1370
(33, -9) .0125
(43, 0) .0792
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The errors for the Mayan square have a median of .0371, a mean of .0548, and a standard deviation of .0487.

The errors for the Aztec diamond have a median of .0573, a mean of .0659, and a standard deviation of .0474.

Discussion

The two means differ by 23.42% of the smaller standard deviation. Thus it is valid to infer that the errors in inferred bias for the two regions are statistically close. Notice further that the average errors in the Mayan square are less than the errors in the Aztec diamond, though it is unclear whether this is statistically significant. Since the effects of finite-sized regions and random sampling should be about the same in both regions, this experiment provides evidence that the Mayan square satisifies the Universality Conjecture as well as the Aztec diamond does—that is, exactly.

References

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