# An Alternative Technique for Proving the Aztec Diamond Theorem and Other Applications

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**Abstract.** A new technique called the superimposition technique is used to prove various combinatorial identities, including the Aztec Diamond Theorem. The technique involves superimposing matchings of a graph and a smaller subgraph, and then partitioning the united graph again into matchings of two subgraphs. Applications of the technique include weighted Aztec diamonds, holey Aztec rectangles, and placement probabilities of dominoes.

**1. Introduction.** The *Aztec diamond* of order $n$ is defined as the union of all unit squares whose corners are lattice points which lie within the region $\{(x, y) : |x| + |y| \leq n + 1\}$. A *domino* is simply a 1-by-2 or 2-by-1 rectangle whose corners are lattice points. A *domino tiling* of a region $R$ is a set of non-overlapping dominoes whose union is $R$. Figure 1-1 shows an Aztec diamond of order 4 and a sample domino tiling.

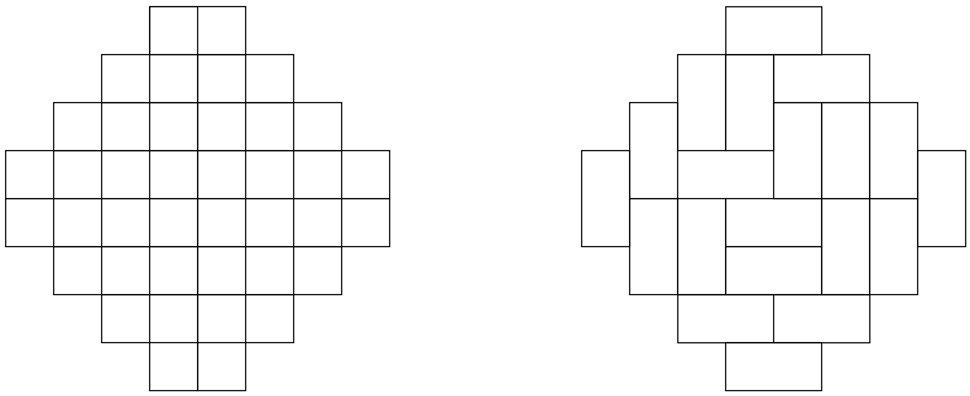

**Figure 1-1.** Order 4 Aztec diamond and a sample domino tiling.

One of the first problems dealing with Aztec diamonds is to count the possible tilings of an Aztec diamond, and to find an explicit formula. In [**1**], it is proved that the number of tilings for the order-$n$ Aztec diamond is $2^{n(n+1)/2}$. The proof introduces concepts such as height functions and alternating-sign matrices.

A related problem proposed in a collection of open combinatorial problems [**4**, Problem 9] is to compute the number of tilings for *"holey Aztec rectangles."* A holey Aztec rectangle looks much like an Aztec diamond, except that the boundary of an $n$-by-$(n+1)$ holey Aztec rectangle consists of diagonals of length $n$, $n+1$, $n$, and $n+1$. In addition, to maintain the balance of squares of different parity so that the region can be tiled, a square is removed from its interior. Figure 1-2 shows the 4-by-5 holey Aztec rectangle with a hole in the center. Note that if $n$ is even, the center square can be the removed square. If $n$ is odd, we cannot remove the center square, but we can remove a square adjacent to it.

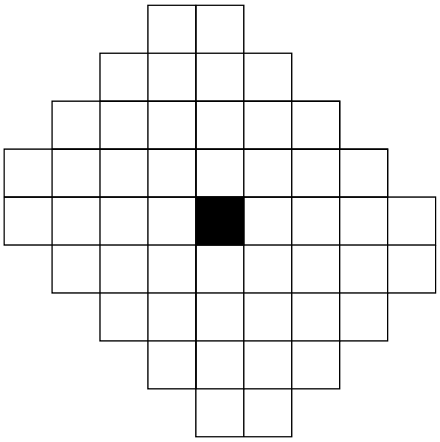

**Figure 1-2.** 4-by-5 holey Aztec rectangle.

As the author went about trying to solve the holey Aztec rectangle problem, he discovered a new combinatorial argument, which has some far-reaching applications to various combinatorial problems. This technique will be called the *superimposition technique.* The technique involves superimposing a matching of a graph onto another, and then partitioning that union into matchings of two other graphs. From the superimposition technique, we can derive an identity that is usually of the form,

$$T(R_1)T(R_2) = T(R_3)T(R_4) + T(R_5)T(R_6),$$

where $T(R_i)$ stands for the number of tilings for a region $R_i$.

In this paper, we show how the superimposition technique can be used to prove the Aztec diamond formula along with several other theorems. In Section 2, we give an alternative proof for counting tilings for the Aztec diamond of order $n$. In Section 3, we discuss weighted matchings and how the recursive formula is useful. Finally, in Section 4, we discuss applications of the technique for regions with holes and for placement probabilities.

**2. Proof of Aztec Diamond Theorem using the superimposition technique.** The order-$n$ *Aztec diamond graph* refers to the graph representation of the order-$n$ Aztec diamond. Throughout this proof, an *Aztec matching* will mean a bipartite matching for an Aztec diamond graph. Figure 2-1 shows the order-4 Aztec diamond graph and an order-4 Aztec matching corresponding to the tiling in Figure 1-1. Thus counting tilings for an Aztec diamond of order $n$ is the same as counting Aztec matchings of order $n$.

To prove that the number of Aztec matchings of order $n$ is $2^{n(n+1)/2}$, we need the following recursive relation.

**Lemma 2-1.** *Let $T(n)$ represent the number of Aztec matchings of order $n$. Then*

$$T(n) = \frac{2(T(n-1))^2}{T(n-2)}.$$

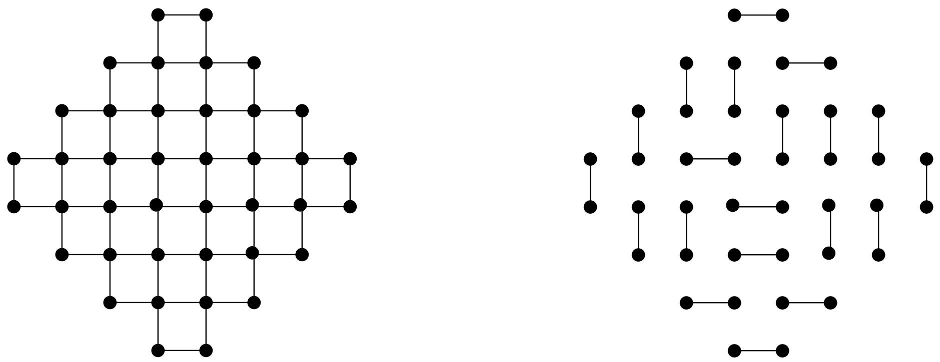

**Figure 2-1.** Left: Order-4 Aztec diamond graph. Right: Order-4 Aztec matching.

*Proof:* It is equivalent to show that

$$T(n)T(n-2) = 2(T(n-1))^2.$$

To prove this relation, we show that the number of ordered pairs $(A, B)$ is twice the number of ordered pairs $(C, D)$, where $A$, $B$, $C$, and $D$ are Aztec matchings of orders $n$, $n-2$, $n-1$, and $n-1$, respectively.

We superimpose an Aztec matching $B$ of order $n-2$ on top of an order-$n$ Aztec matching $A$ so that the matchings are concentric. Figure 2-2 shows Aztec matchings of orders 3 and 5, and the result of superimposing the two matchings. In the combined graph, the white vertices are shared by both the order-3 and order-5 matchings. The black vertices are from the order-5 matching only. Note that some edges are shared by both matchings. Note also that each black vertex has degree 1 in the combined graph, whereas each white vertex has degree 2.

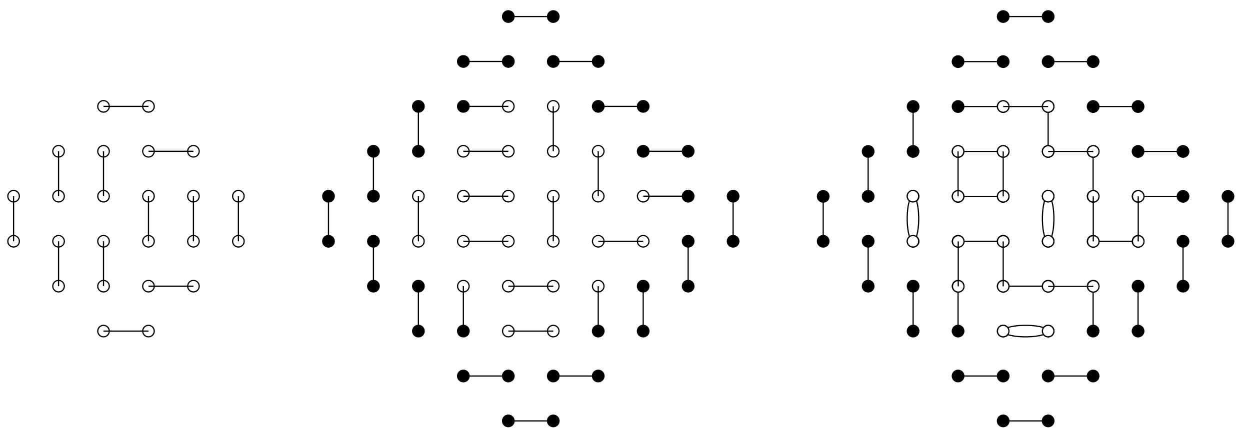

**Figure 2-2.** Left: Order-3 Aztec matching. Center: Order-5 Aztec matching. Right: The two matchings combined.

Now consider overlapping two Aztec matchings of order $n-1$, shown in Figure 2-3. Let us call the left and right matchings $C$ and $D$, respectively. The overlapping can be done in either of two ways. Figure 2-4 shows the two possibile resulting graphs by superimposing matchings $C$ and $D$ in Figure 2-3, as we will now demonstrate.

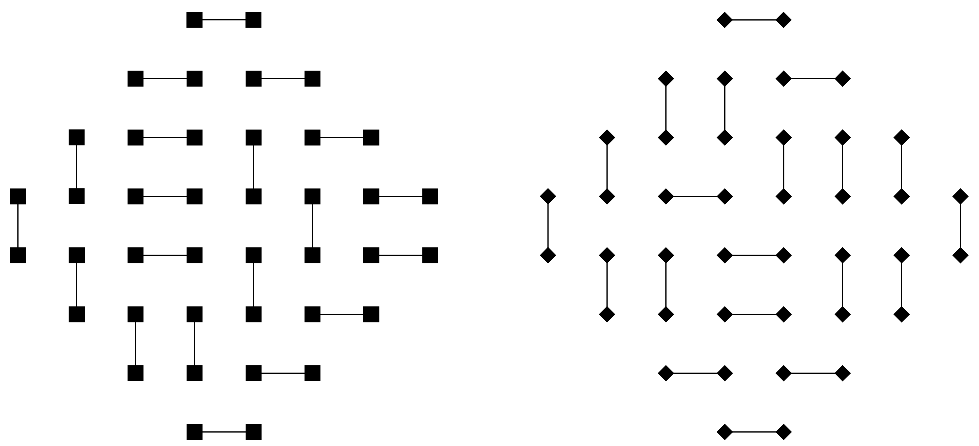

**Figure 2-3.** Aztec matchings $C$ and $D$ of order 4.

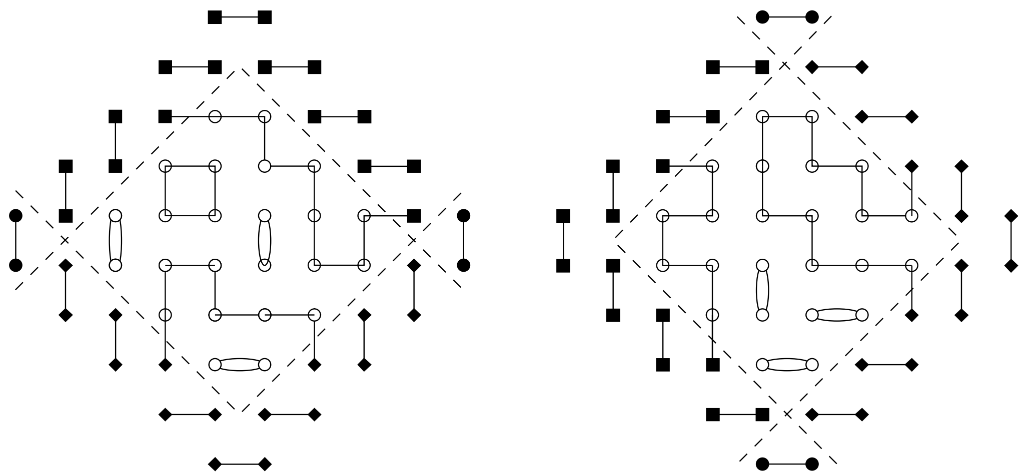

**Figure 2-4.** The two possible combinations of the matchings $C$ and $D$.

Each graph in Figure 2-4 can be partitioned into two order-4 Aztec matchings and two extra segments. The left graph was made by fitting matching $C$ to the top and matching $D$ to the bottom of the order-5 diamond, and then adding two side edges. The graph on the right was made by fitting matching $C$ to the left and matching $D$ to the right of the order-5 diamond, and then adding the top and bottom edges. In both cases, each of the center vertices has degree 2, and all other vertices have degree 1. The graphs resemble order-3 Aztec matchings on top of order-5 Aztec matchings.

In general, we are given a graph $G$ on the vertices of the order-$n$ Aztec diamond graph with the following properties:

1. The inner vertices in $G$ that form an order-$(n-2)$ Aztec diamond have degree 2.
2. The remaining outer vertices in $G$ have degree 1.
3. The edges of $G$ form cycles, doubled edges, single edges, and lattice paths of length greater than 1.

Let us call a graph with such properties a *doubled Aztec graph.* For each superimposition we have described so far, the result is a doubled Aztec graph. We want to show that the number of partitions of $G$ into two Aztec matchings $A$ and $B$ of orders $n$ and $n-2$ is equal to the number of partitions of $G$ into two order-$(n-1)$ Aztec matchings $C$ and $D$ (along with two line segments). We will show that this number is $2^k$, where $k$ is the number of cycles in $G$. Since $G$ is bipartite, all cycles have even length. These cycles are contained in the middle common vertices, as they are the only vertices with degree 2. Each cycle can then be partitioned so that every other edge will go to the same subgraph; adjacent edges go to different subgraphs. For each cycle, there are two ways to decide which half of the cycle goes to $A$ or $B$. Similarly, there are two ways to decide which half goes to $C$ or $D$. All doubled edges in $G$ are split and shared by each subgraph. It remains to show that the other edges must be partitioned uniquely.

Figure 2-5 shows the degree-one vertices labeled T, U, V, W, X, and Y. Each vertex labeled Y will match with exactly one vertex labeled T, U, V, or W. For each label T, U, V, and W, exactly one vertex will not be connected to a Y-vertex. We denote these special vertices $T'$, $U'$, $V'$, and $W'$. Those vertices must be connected by some path. There are three possible kinds of "paths" from $T'$ to $U'$:

1. A path that goes through the interior O-vertices.
2. A path of length 1 joining $T'$ to $U'$, if $T'$ is adjacent to $U'$.
3. A path of two disjoint segments, in which $T'$ is adjacent to $U'$, and $T'$ is connected to the X-vertex directly above, forcing the other X-vertex at the top to connect downwards to $U'$.

Let us call a path from either of the first two types a *continuous path.* Let us also call the third type a *path through the X-vertices.*

Let us see why these are the only kinds of paths that may exist from $T'$ to a $U'$. The vertex $T'$ must join an O-, U-, X-, or V-vertex. If $T'$ is connected to an O-vertex, then the path must continue through the O-vertices until it reaches a U-, V-, or W-vertex. Clearly, the path must stop at $U'$. If $T'$ is connected directly to the U-vertex, then we already have a path from $T'$ to $U'$. If $T'$ connected to the top X-vertex above it, then we get a path through the X-vertices. However, if $T'$ is connected to the X-vertex to the left, the other X-vertex is forced to join $V'$; therefore a path from T to U is impossible. Finally, $T'$ cannot be connected to the V-vertex, for then there is a path from $T'$ to $V'$.

In a doubled Aztec graph, there must be paths joining $T'$ to the $U'$ and $V'$ to $W'$, or paths joining $T'$ to $V'$ and $U'$ to $W'$. However, we cannot have paths going from $T'$ to $W'$

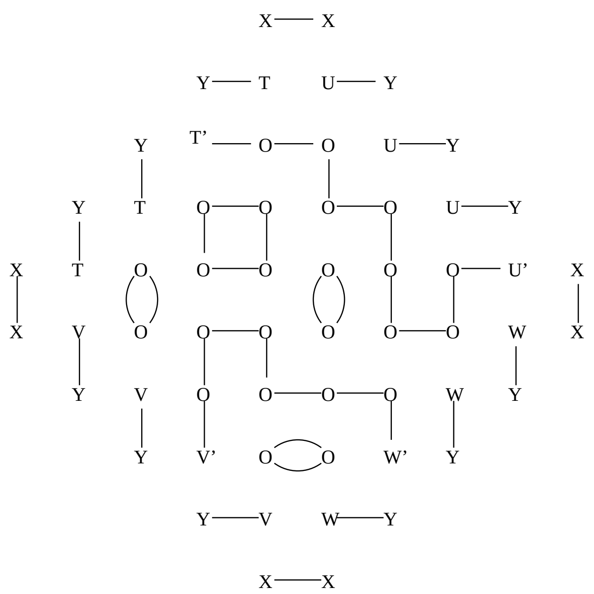


**Figure 2-5.** Specially labeled graph of doubled Aztec graph $G$.

and from $U'$ to $V'$. If such paths existed, then both paths must travel through the O-vertices and intersect, thus forcing the degree of some O-vertex to be more than 2.

Let us show that the segments from both ends of a path must belong to the same subgraph in any partition of $G$. Let us 2-color the vertices of $G$ black and white so that black vertices are adjacent to white vertices. The T- and W- vertices must be the same color; let us color all the T- and W-vertices white. Then the U- and V-vertices must be of the other color, which is black. Therefore any continuous path from T to U, from T to V, from U to W, or from V to W must have odd length since the path goes from a black to a white vertex. Thus the segments from both ends of a continuous path must belong to the same subgraph in any partition of $G$. The previous statement also holds true for paths through the X-vertices. In any partition of $G$, the two X-vertices must be in the same subgraph.

Thus, when we partition $G$ into matchings $A$ and $B$ of orders $n$ and $n-2$, we must always place the ending segments into $A$ and determine the rest of the partition thereafter. Such a partition always exists.

Without loss of generality, let the paths in $G$ connect $T'$ to $U'$ and $V'$ to $W'$. When $G$ is partitioned into two matchings $C$ and $D$, both of order $n-1$, one matching (say $C$) must have both $T'$ and $U'$, as they are the ends of the same lattice path. The other matching $D$ must have $V'$ and $W'$. The paths are then partitioned uniquely. Thus we can partition $G$ into two order-$(n-1)$ Aztec matchings placed top-bottom such that the top diamond contains the T- and U-vertices, and the bottom diamond contains the V- and W-vertices. However, it is not possible to partition $G$ into two side-by-side Aztec matchings of order $n-1$ such that one conains the T- and V-vertices, and the other contains the U- and W-vertices. The reason is this: since the left matching has $V'$, it would then contain $W'$. The latter cannot happen, since $W'$ is in the other matching.

Hence each doubled Aztec graph can be partitioned into two order-$(n-1)$ Aztec matchings in one way (top-bottom) or the other (left-right), but never both. The partition of the paths is uniquely determined.

The number of ways to combine Aztec matchings of orders $n$ and $n-2$ is $T(n)T(n-2)$, while the number of ways to combine two order-$(n-1)$ matchings is $2(T(n-1))^2$. Each combination becomes a doubled Aztec graph, so the relation is proved. $\square$

Having proved the recursive relation, we can now compute the number of tilings of an Aztec diamond of order $n$.

**Theorem 2-2.** *The number of tilings of the order-$n$ Aztec diamond is* $2^{n(n+1)/2}$.

*Proof:* We can verify that there are 2 tilings for the order-1 Aztec diamond, and that there are 8 tilings for the order-2 Aztec diamond. Indeed, $2^{1(1+1)/2} = 2$ and $2^{2(2+1)/2} = 8$, so we have our two base cases. Now we presume that the formula holds for $n = k-1$ and $n = k$. Using Lemma 2-1, we calculate the number of tilings for order $k+1$:

$$\begin{aligned} T(k+1) &= \frac{2(T(k))^2}{T(k-1)} = \frac{2(2^k(k+1)/2)^2}{2^k(k-1)/2} \\ &= 2^{1+k(k+1)-k(k-1)/2} = 2^{(k^2+3k+2)/2} = 2^{(k+1)(k+2)/2}. \end{aligned}$$

Thus the number of order-$n$ Aztec matchings is $2^{n(n+1)/2}$. $\square$

**3. Formula for weighted Aztec diamonds.** Given a graph $G$, we can assign a weight to each edge to form a *weighted graph*. The *weight of a graph $G$* is the product of the weights of all the edges in $G$. We denote the weight of $G$ by $w(G)$. Similarly, the *weight of a matching* on $G$ is the product of the weights of each edge in that matching. We also define the *weighted sum* of $G$ to be the sum of the weights of all possible matchings on $G$. We let $W(G)$ stand for the weighted sum of $G$.

Consider a weighted Aztec diamond graph $A$ of order $n$. Define $A_{top}$ to be the upper order-$(n-1)$ Aztec sub-diamond along with its corresponding edge weights in $A$. Similarly, we can refer to the bottom, left, and right subgraphs of $A$ as $A_{bottom}$, $A_{left}$, and $A_{right}$, which are all order $n-1$ Aztec sub-diamonds. Finally, let $A_{middle}$ be the inner order-$(n-2)$ Aztec diamond within $A$. Figure 3-1 shows an Aztec diamond and its five sub-diamond graphs.

It turns out that the superimposition technique can also be used to establish an identity for weighted Aztec diamond graphs. The following theorem shows how the weighted sum of a weighted Aztec diamond can be expressed in terms of the weighted sums of the subdiamonds and a few edge weights.

**Theorem 3-1.** *Let $A$ be a weighted Aztec diamond of order $n$. Also let $t$, $b$, $l$, and $r$ be the weights of the top, bottom, left, and right edges of $A$, respectively. Then*

$$W(A) = \frac{l \cdot r \cdot W(A_{top}) \cdot W(A_{bottom}) + t \cdot b \cdot W(A_{left}) \cdot W(A_{right})}{W(A_{middle})}.$$

*Proof:* This proof is very similar to Lemma 2-1, except that we must fill in the details concerning the weights. Indeed, we want to show that

$$W(A) \cdot W(A_{middle}) = l \cdot r \cdot W(A_{top}) \cdot W(A_{bottom}) + t \cdot b \cdot W(A_{left}) \cdot W(A_{right}). \tag{3-1}$$

We have seen how a doubled Aztec graph $G$ of order $n$ can be decomposed into subgraphs in two of three following ways:

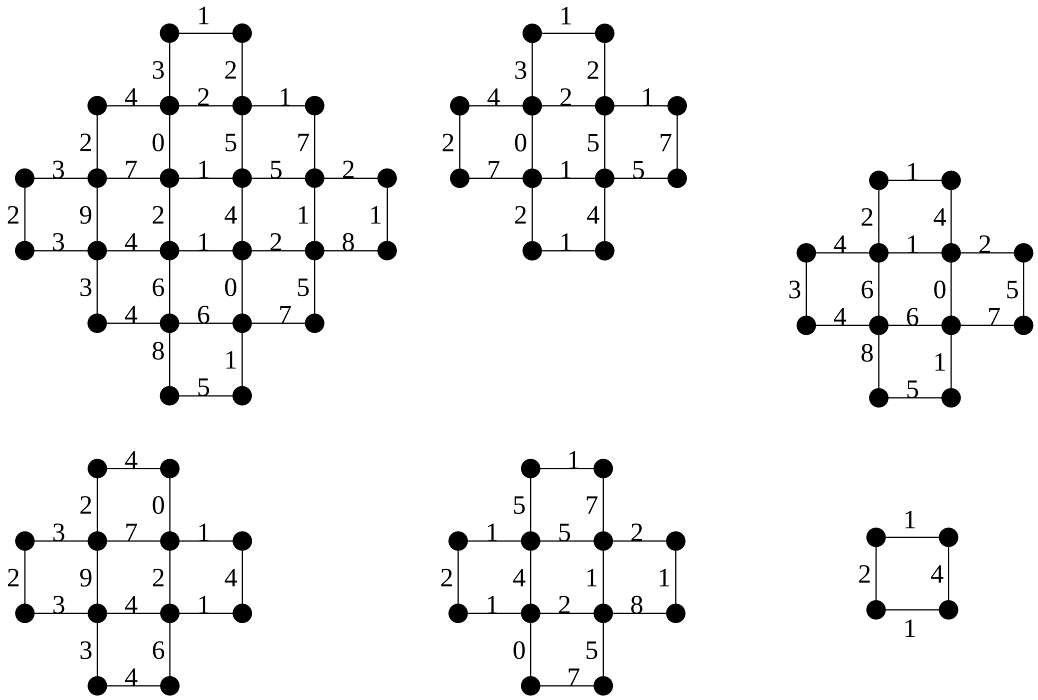

**Figure 3-1.** Top Row: A weighted Aztec diamond graph $A$, followed by $A_{top}$ and $A_{bottom}$. Bottom Row: $A_{left}$, $A_{right}$, and $A_{middle}$.

1. (Big-small) Two Aztec matchings of orders $n$ and $n-2$.
2. (Top-bottom) Top and bottom Aztec matchings of order $n-1$, plus the left and right edges.
3. (Left-right) Left and right Aztec matchings of order $n-1$, plus the top and bottom edges.

As we know, $G$ can always be decomposed via Big-small and by either Top-bottom or Left-right (but not both). The number of possible decompositions by either method is $2^k$, where $k$ is the number of cycles in $G$.

Each edge in $G$ becomes a part of exactly one of the subgraphs. Therefore, the product of the weights of the subgraphs will always equal to the weight of $G$, since each edge weight is multiplied once.

Recall that $W(A)$ is the sum of the weights of all possible matchings on $A$. Then

$$W(A) \cdot W(A_{middle}) = \sum_G 2^{k(G)} w(G)$$

where $G$ ranges over all doubled Aztec graphs of order $n$, and $k(G)$ is the number of cycles in $G$. Each term in the sum represents the weight of $G$ multiplied by the number of ways to partition $G$ via Big-small. Each partition is accounted for in $W(A)W(A_{middle})$. Similarly, we also have

$$l \cdot r \cdot W(A_{top}) \cdot W(A_{bottom}) + t \cdot b \cdot W(A_{left}) \cdot W(A_{right}) = \sum_G 2^{k(G)} w(G)$$

Thus both sides of Equation 3-1 are equal to a common third quantity, so the relation is proved. □

A special case of Theorem 3-1 is Lemma 2-1, the inductive step in counting tilings for Aztec diamonds. We achieve that result simply by assigning 1 as the weight of each edge. Then the weight of each matching is 1, so the weighted sum is simply the number of Aztec matchings.

Theorem 3-1 may be used to find the weighted sum of a *fortress-weighted Aztec diamond*. Imagine rotating an Aztec diamond graph by 45 degrees and then partitioning the edges of the graph into *cells*, or sets of four edges forming a cycle. In a fortress-weighted Aztec diamond, there are two types of cells: (1) cells whose edges are weight 1, and (2) cells whose edges are weight 1/2. Cells with edge-weights of 1/2 are adjacent to cells with edge-weights of 1. (See Figure 3-2).

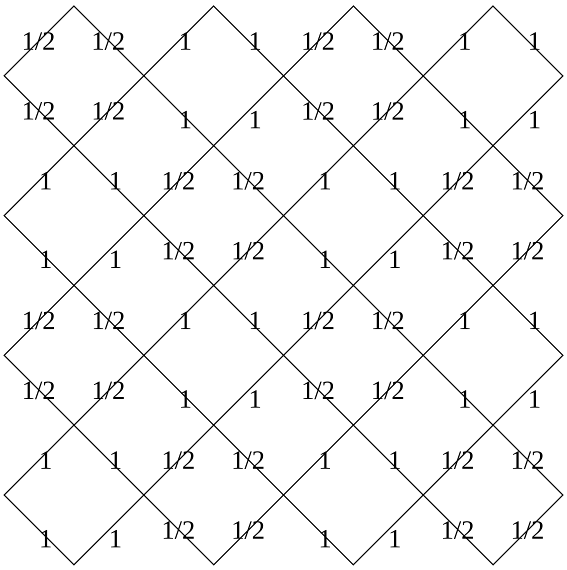

**Figure 3-2.** A fortress-weighted Aztec diamond, rotated by 45 degrees.

There are three kinds of fortress-weighted Aztec diamonds:

1. The order $n$ is odd, and all edges in the corner cells have weight 1.

2. The order $n$ is odd, and all edges in the corner cells have weight 1/2.

3. The order $n$ is even, and two opposite corners have edges weighted 1/2, and the other two corners have edges weighted 1.

Let $A_n$, $B_n$, and $C_n$ stand for the weighted sums of the these diamonds, respectively. We then use Theorem 3-1 to establish relations among $A_n$, $B_n$, and $C_n$. They are:

$$A_{2k+1} = \frac{1 \cdot 1 \cdot C_{2k} \cdot C_{2k} + 1 \cdot 1 \cdot C_{2k} \cdot C_{2k}}{A_{2k-1}} = \frac{2{C_{2k}}^2}{A_{2k-1}},$$

$$
\begin{aligned}
B_{2k+1} &= \frac{\frac{1}{2}\cdot\frac{1}{2}\cdot C_{2k}\cdot C_{2k}+\frac{1}{2}\cdot\frac{1}{2}\cdot C_{2k}\cdot C_{2k}}{B_{2k-1}} = \frac{\frac{1}{2}{C_{2k}}^2}{B_{2k-1}},\\
C_{2k} &= \frac{\frac{1}{2}\cdot\frac{1}{2}\cdot A_{2k-1}\cdot A_{2k-1}+1\cdot 1\cdot B_{2k-1}\cdot B_{2k-1}}{C_{2k-2}} = \frac{\frac{1}{4}{A_{2k-1}}^2+{B_{2k-1}}^2}{C_{2k-2}}.
\end{aligned}
$$

From these relations, we can easily prove by induction that for odd $k$,

$$
\begin{aligned}
A_{2k+1} &= (5/4)^{k(k+1)},\\
B_{2k+1} &= (5/4)^{k(k+1)},\\
C_{2k} &= (5/4)^{k^2}.
\end{aligned}
$$

For even $k$,

$$
\begin{aligned}
A_{2k+1} &= 2(5/4)^{k(k+1)},\\
B_{2k+1} &= \frac{(5/4)^{k(k+1)}}{2},\\
C_{2k} &= (5/4)^{k^2}.
\end{aligned}
$$

The importance of the fortress-weighted Aztec diamond comes from the problem of computing the number of diabolo tilings for a fortress. A *diabolo* is either an isoceles right triangle or a square, formed by joining two smaller isoceles right triangles. A *fortress* is a diamond shaped region that is made up of isosceles right triangles and can be tiled by diaboloes. A fortress and a sample tiling by diaboloes are shown in Figure 3-3.

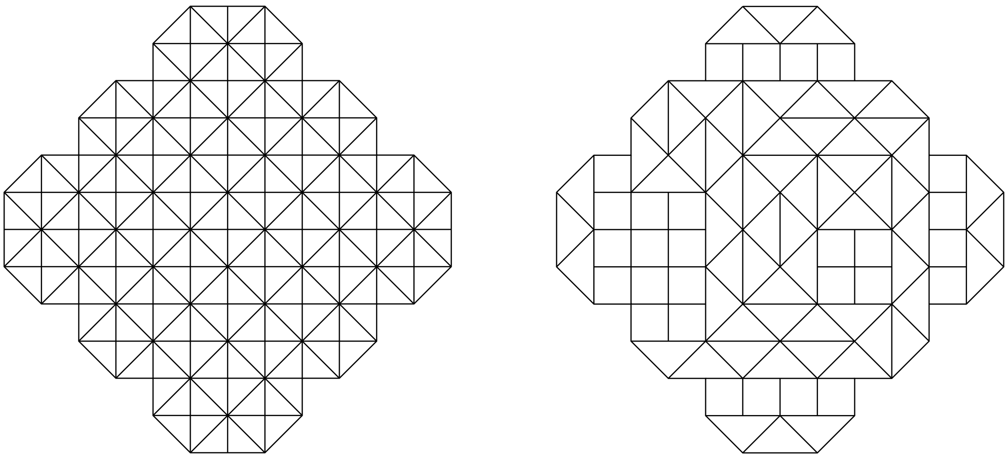

**Figure 3-3.** A fortress and sample diabolo tiling.

To transform a fortress graph into a weighted Aztec diamond graph, we must use a method called *urban renewal.* This technique is explained in [**5**] along with proofs and applications. On page 27 of [**5**], the transformation is described for the fortress, and the number of tilings for the fortress would be the weighted sum of the fortress-weighted diamond times some power of 2.

**4. Applications to regions with holes.** Let us label some the squares in an Aztec rectangle as shown in Figure 4-1. We label a square only if the region becomes tileable after deleting that square. We let $R_{n,a,b}$ represent the $n$-by-$(n-1)$ Aztec rectangle whose square $(a, b)$ has been deleted. Then we can apply our technique and come up with a theorem which relates the numbers of tilings among holey Aztec Rectangles.

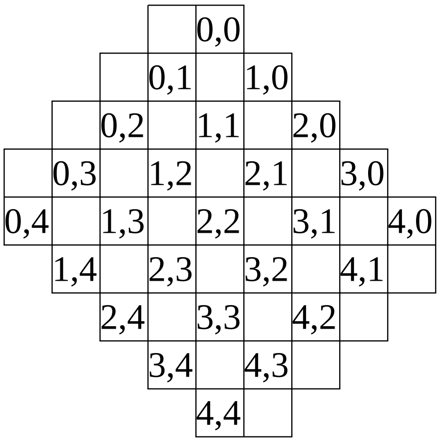


**Figure 4-1.** Labeling of squares in a holey Aztec rectangle. By removing square $(a, b)$ we obtain $R_{4,a,b}$.

**Theorem 4-1.** *Let $T(R)$ stand for the number of tilings for region $R$. Then for $a$, $b$ between 1 and $n-1$, the number of tilings of $R_{n,a,b}$ is expressed in the follwing relation:*

$$T(R_{n,a,b}) = \frac{T(R_{n-1,a,b-1})T(R_{n-1,a-1,b}) + T(R_{n-1,a,b})T(R_{n-1,a-1,b-1})}{T(R_{n-2,a-1,b-1})}.$$

The proof is very similar to that of Lemma 2-1. Instead of superimposing an order-$n$ Aztec matching on an order-$(n-2)$ Aztec matching, we superimpose $R_{n-2,a-1,b-1}$ on top of $R_{n,a,b}$ so that the holes align to the same spot. Given the graph $G$ resulting from the superimposition, we can partition it into two $(n-1)$-by-$n$ holey Aztec rectangles. The partition can be done either left-right or top-bottom, but only one or the other. The left-right rectangles are isomorphic to $R_{n-1,a-1,b}$ and $R_{n-1,a,b-1}$. The top-bottom rectangles are isomorphic to $R_{n-1,a,b}$ and $R_{n-1,a-1,b-1}$.

We can also use the technique to get recurrence relations for placement probabilities of dominoes in tilings of Aztec diamonds. Let domino $D$ be a specified pair of adjacent squares in an Aztec diamond. The *placement probability* of $D$ in an order-$n$ Aztec diamond is the probability that $D$ will appear in a tiling of the order-$n$ Aztec diamond, given that all tilings are equally likely.

Placement probabilities are of interest in the study of random tilings. If we look at a random tiling of an Aztec diamond of large order, we notice four regions in which the dominoes form a brickwork pattern, and a central circular region where dominoes are mixed up. The placement probability of any domino at the center of the diamond will be 1/4. However, in the top corner, dominoes which conform to the brickwork will have probabilities near 1. All other dominoes in this corner would have probabilities near 0. More about placement probabilities can be read in [**3**] and [**2**].

We could calculate the placement probability of a domino with the following steps. First, we replace the domino with a two-square hole in the Aztec diamond. Then we compute the number of tilings of that diamond with the hole. Finally, we divide it by the number of tilings of the (complete) Aztec diamond.

We can express the number of tilings of the order-$n$ Aztec diamond with the hole at $D$ in terms of tilings of lower-order Aztec diamonds with holes. But first, let us introduce some notation. We will let $A_n - D$ stand for the order-$n$ Aztec diamond with domino $D$ missing. The dominoes $D_{up}$, $D_{down}$, $D_{left}$, and $D_{right}$ will represent dominoes shifted up, down, left, and right by a square relative to $D$ in the Aztec diamond. Then $A_{n-1} - D_{up}$ is the order-$(n-1)$ diamond such that when it is placed concentrically with $A_n - D$, the hole of $A_{n-1} - D_{up}$ will match up with $D_{up}$. The regions $A_{n-1} - D_{down}$ and so forth represent similar Aztec diamonds with domino holes. Finally, $A_{n-2} - D$ is the order-$(n-2)$ Aztec diamond such that when $A_{n-2} - D$ is placed right on $A_n - D$, domino $D$ is missing. See Figure 4-2 for examples.

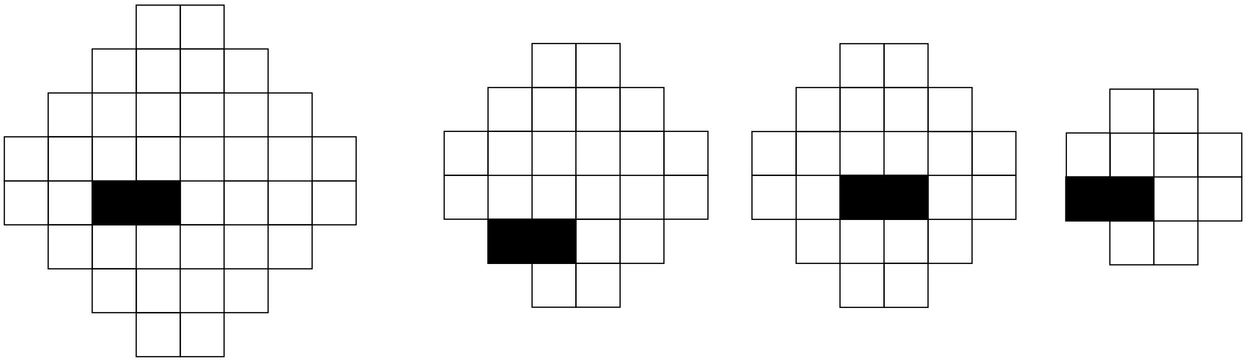

**Figure 4-2.** Regions from left: $A_n - D$, $A_{n-1} - D_{down}$, $A_{n-1} - D_{right}$, $A_{n-2} - D$.

By using the superimposition technique, we can relate the number of tilings of these Aztec diamonds with holes:

$$T(A_n - D)T(A_{n-2} - D) = T(A_{n-1} - D_{up})T(A_{n-1} - D_{down}) + T(A_{n-1} - D_{left})T(A_{n-1} - D_{right}).$$

We also have the following relation, which relates numbers of tilings of Aztec diamonds:

$$T(A_n)T(A_{n-2}) = 2T(A_{n-1})T(A_{n-1}).$$

We can then derive a relation among placement probabilities of dominoes in Aztec diamonds of orders $n$, $n-1$, and $n-2$. When we divide the first relation by the second, we get

$$P(A_n, D)P(A_{n-2}, D) = \frac{P(A_{n-1}, D_{up})P(A_{n-1}, D_{down}) + P(A_{n-1}, D_{left})P(A_{n-1}, D_{right})}{2},$$

where $P(R, D)$ is the placement probability on domino $D$ in region $R$. The probability $P(R, D)$ was computed by dividing $T(R - D)$ by $T(R)$.

## References


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