# An Illustrative Study of the Enumeration of Tilings:

Conjecture Discovery and Proof Techniques

**by Chris Douglas**

## 1.0 Introduction

We have exact formulas for the number of tilings of only a small number of regions: Aztec Diamonds, Fortresses, and regions composed of Lozenge Tiles. However, these regions, particularly aztec diamonds and lozenge tilings are fundamental, and numerous other regions types can be reduced to weighted versions of these graphs using urban renewal techniques. This paper will illustrate the process by which you can generate conjectures about new region types and then prove those conjectures using various graph manipulation techniques and combinatorial arguments to reduce them to previously known results, in particular to the case of weighted aztec diamonds. The example used throughout is a conjecture involving tilings of the plane using triangles and squares, originally proposed in [5] and proved in [3]. We will follow the conjecture from the beginning, using it to illustrate the process of the choice of the boundary conditions determining a finite region, computer experimentation, conjecture formation, and proof, primarily by means of urban renewal and a related transformation. The proof which is developed here is by no means the shortest or smoothest version; it is rather presented in the natural structure of its discovery in an attempt to elucidate the process.

## 2.0 Conjecture Formation

### 2.1 Region Generation

We begin the process with an infinite tiling of the plane and would like to choose a finite subregion which will be an appropriate analog of either an aztec diamond or a lozenge hexagon for this type of tiling (in other words, a region which has a combinatorially pleasing formula which is likely to be an outgrowth of weighted versions of these graphs). For example, let us begin with the following tiling, which because of the symmetries of the graph, we will explore in the context of a reduction to weighted aztec diamonds:

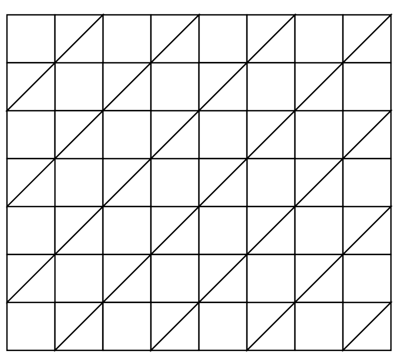


**FIGURE 1. An Infinite Tiling of the Plane**

To elucidate the structure of the region, we shade half of the tiles in the graph such that every shaded tile is bordered by unshaded tiles and vice versa. The primary method for choosing a finite subregion is as follows: First choose an even number of anchor points, the number depending on the symmetry of the graph. Second, beginning at one anchor point, move toward the next point so that only a single type of tile (shaded or unshaded) runs along the boundary, and switch the type of boundary tile at each anchor point until the region is complete. In the example, we choose four anchor points, in a diamond pattern, and carve the region as follows (we could also put the anchor points where six tiles meet instead of four tiles, and explore the resulting different region, but we will focus on this particular case):

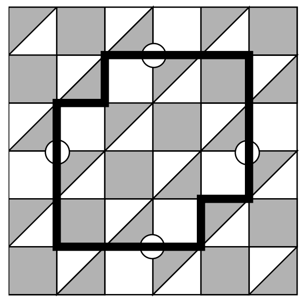


**FIGURE 2. The Shaded Tiling showing Anchor Points and the Aztec Boundary of Order 4**

A more fundamental, though in this case unnecessary, alternative for choosing a region utilizes pure phases of the graph to determine the natural region. Briefly, one finds all pure phases of a graph, otherwise known as the extremal tilings, as determined by an appropriate height function, and fits these extremal tilings together, determining the boundary by the natural structure of these phases.

### 2.2 Experimentation

At this point we would like to count the tilings of this region, and larger analogous regions, in an attempt to find a general formula. Toward that end, we first convert the graph into its dual picture, where a pairing of vertices corresponds to a domino in the original graph. Thus we have:

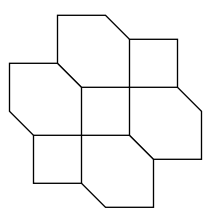
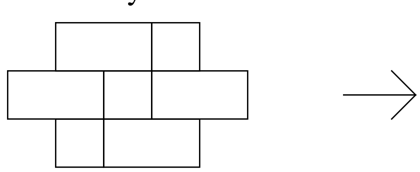


**FIGURE 3. The Dual Picture for the Graph of Order 4**

To insert this graph into the computer, we need to manipulate it into a square grid form. In this case we can proceed immediately to:



XVAX
XVAXVA
VAXVAX
XVAX

**FIGURE 4. Converting the Dual Picture to a Vax File Representation**

In this representation, X's represent vertices which can be matched in any direction, V's vertices which can be matched any direction except down, and A's vertices which can be matched any direction except up. In general, manipulating a graph into a square grid format may not be so easy. For example, if a vertex of order six or more appears in the graph, we need to perform 'vertex splitting' as follows:

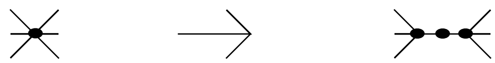


**FIGURE 5. Vertex Splitting**

This operation does not affect either the number of matchings or the weighting on individual matchings. As is sometimes also necessary, we can do this splitting even when a vertex is attached to only two or four other vertices.

At this stage we have a so called Vax file, composed of V's, A's, and X's in the appropriate pattern. We can now compute the number of tilings of this region and its larger analogs using either vaxmaple, a program written by James Propp and David Wilson based on ideas by Greg Kuperberg and David Wilson, or vaxmacs, an extension of vaxmaple written by David Wilson.

### 2.3 Conjecture Formation

After computing the number of tilings based on a square grid region of side length 4n for n=1,2, and 3, we have the sequence 16, 4096, 16777216. We can now, as James Propp did in [5], form the following:

**Conjecture:** The region of order 4n has $2^{2n(n+1)}$ tiling:

If the pattern is complicated, more sophisticated means may be needed to discover the pattern. Many of these techniques are described in [1].

## 3.0 Proof Techniques

### 3.1 Urban Renewal

Urban renewal is a graph manipulation technique which preserves, or alters by a constant factor, the number of matchings of the graph. The standard technique is the following conversion, which decreases the number of matchings to half its original value (Circled vertices are permitted to have connections with other vertices in the larger graph, and these connections are unaffected by the transformation):

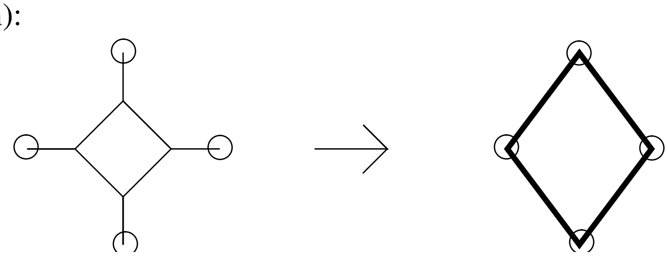


**FIGURE 6. The Urban Renewal Transformation**

Thick lines represent half weighted edges - any matching containing a half weighted edge counts as only half a matching (or one quarter if it has two half weightings, et cetera).

A slight variation, which arises as a simple corollary of the main urban renewal transformation, is also useful for the example at hand (Again, circled vertices are permitted connections outside of the shown graph):

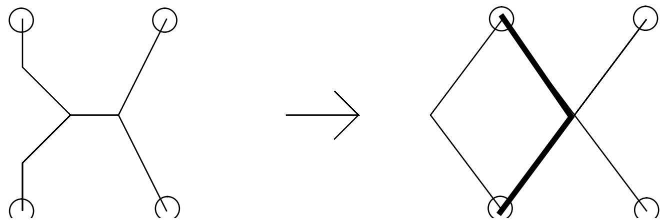


**FIGURE 7. A Second Urban Renewal Technique**

This conversion does not change the number of matchings by a constant factor as did the previous conversion.

Let us return to our example, now in the order eight (n=2) case:

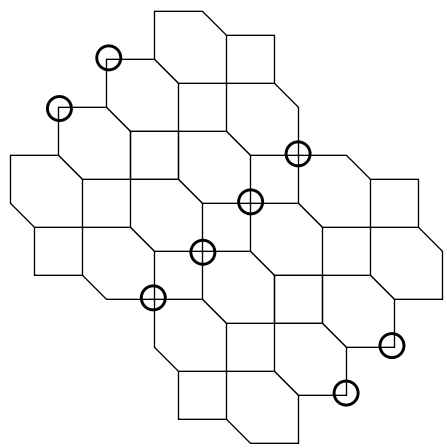


**FIGURE 8. The Order Eight Graph with Vertices Marked to be Split**

In order to apply urban renewal to convert the graph into a weighted aztec diamond, perform vertex spliting on the eight indicated vertices, and then apply both forms of urban renewal, three of the first kind and eight of the second kind. This leaves us with:

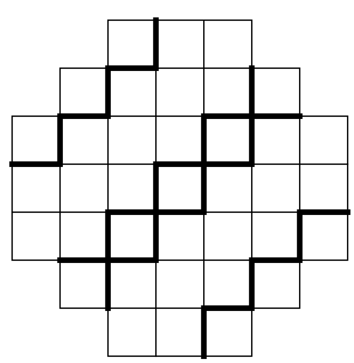


**FIGURE 9. The Order Eight Graph after Urban Renewal**

## 3.2 Specific Case Analysis

We now have a graph which closely resembles a weighted aztec diamond. In general, at this point, the objective is to reduce the weighting scheme to an unweighted aztec diamond, or one with a systematic weighting pattern covering the entire graph. This reduction is done primarily by means of determining groups of edges which cannot be matched together or must be matched together and removing the weightings on these edges in the individual cases while multiplying the number of matchings by a constant factor. If the graph is not an immediately obvious corollary of one of these forms, the next logical step is to analyze a small specific case to gain a better understand of any divisions or patterns which might assist in the general case.

In this case, writing out the possibilities for the specific graph of n=2 is very instructive. However, the patterns are straightforward enough that we will proceed directly to the general case.

## 3.3 The Generalized Analysis

The first step in the final analysis of the problem is performing the urban renewal sequence in a general setting to determine the exact constant factor by which you need to multiply the number of matchings. Consider a region from our example of order 4n. The dual graph will have rows of hexagons and squares in the following sequence: 2n hexagons, 2n+1 squares, 2n hexagons, 2n-1 squares, 2n hexagons, 2n+1 squares, ..., 2n+1 squares, 2n hexagons. There are a total of 4n-1 rows. Now, we performed urban renewal along the short rows of squares, of which there are n-1, consisting of 2n-1 urban renewals per row. So we must adjust the number of matchings by a factor of $2^{(2n-1)(n-1)} = 2^{2n^2-3n+1}$. The other urban renewal techniques used on the edges do not affect the number of matchings.

Now we are dealing only with a weighted aztec diamond pattern - in this case an aztec diamond with the four corner vertex pairs matched, which we will refer to as a Modified Aztec Diamond. We can eliminate most of the half weighted edges by noting that with the exceptions of the patterns at the northwest and southeast sides, the edges occur in groups of four, surrounding a vertex and so can be removed by multiplying the number of matchings by one half for each vertex. There are n-1 rows with 2n vertices per row, so we modify the matching number by a factor of $2^{2n-2n^2}$.

We now have reduced as far as is possible using these methods. We are left with two strings of half weighted edges, one on each side. We cannot eliminate these edges, as it is not a given that any of them are or aren't matched. At this point we employ the next technique: case division. It is clear that at most one weighted edge on each side can be matched, so we have precisely three cases: neither side contains a half weighted matching, one side does, or both sides do.

The case where neither side has weighted matchings is easy - there are two ways to match the outside ring and in the center is left an unweighted aztec diamond of order 2n-1:

$$N_0(4n) = 2(AZ(2n-1))$$

For the other two cases, there are a number of ways to proceed at this point: the easy way, the hard way, and the really hard way. Because it illustrates some useful combinatorial techniques, we will go about the proof the hard way, with a brief excursion to glimpse the really hard way, and afterwards note how the arguments could be simplified.

If one side contains a matched edge, we can simplify matters by noting that both sides are completely equivalent, and half of the 4n-2 edges on each side are equivalent to the other half on that side, as their matchings are realized by a single block rotation from one another (it is the two half weighted edges sharing the same vertex which are related in this way). Thus, in fact, we have only 2n-1 cases. If we match the i'th vertex (counting along the 2n-1 vertices set in from the edge on each side, numbering from lower left to upper right on both edges) along one of its two half weighted neighbors, then the entire side is forced. There are now two possibilities for matching the other side so that none of the half weighted edges are matched. The first possibility forces everything except an aztec rectangle of order 2n by 2n-1 with the (i-1)'th vertex (of the outer edge) removed from the long side, and the second possibility forces everything except an aztec rectangle of order 2n by 2n-1 with the i'th vertex removed.

At this point we bring in another necessary and simplifying technique - the introduction of notation. Define P(n,m,r,s) to be the number of matchings of an n by m aztec rectangle with the r'th vertex removed from one long side and the s'th vertex removed from the other long side, and P(n,m,r) to be the number of matchings of an n by m aztec rectangle with the r'th vertex removed from one long side. (We number the vertices beginning at 0, not 1.) Thus, for example, P(5,3,3,1) corresponds to:

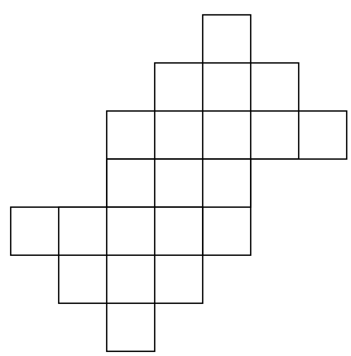


**FIGURE 10. A 5 by 3 Aztec Rectangle with the 1st and 3rd Vertices Removed from Opposite Sides**

We note that

$$P(n, n-1, 0) = AZ(n-1)$$

$$P(n, n-1, r) = P(n, n-1, n-r-1)$$

$$P(n, n-2, r, s) = P(n, n-2, s, r)$$

Using this notation, we can write the number of matchings with one half weighted vertex as:

$$N_1(4n) = 4\sum_{i=1}^{2n-1}(P(2n, 2n-1, i-1) + P(2n, 2n-1, i))$$

We can similarly write down the number of matchings including two half weighted vertices in terms of aztec rectangles. Briefly, given any pair of vertices, one on each side, to be matched to a half weighted edge, there are two possibilities on each side, a total of four combinations, and both sides are then forced, leaving an aztec rectangle of order 2n+1 by 2n-1 with a vertex removed from each side. There are 2n-1 vertices on each side and so the total number of matchings in this case is:

$$N_2(4n) = 4\sum_{i=1}^{2n-1}\sum_{j=1}^{2n-1} P(2n+1, 2n-1, i, j)$$

We can now put the urban renewals, the half weighted vertices, and the three cases together to yield the following formula:

$$\begin{aligned}N(4n) &= 2^{2n^2-3n+1}2^{-(2n^2-2n)}\Bigg((2P(2n, 2n-1, 0)) \\ &+ 2^{-1}\left(4\sum_{i=1}^{2n-1}(P(2n, 2n-1, i-1) + P(2n, 2n-1, i))\right) \\ &+ 2^{-2}\left(4\sum_{i=1}^{2n-1}\sum_{j=1}^{2n-1} P(2n+1, 2n-1, i, j)\right)\Bigg) \\ &= 2^{1-n}\Bigg(2P(2n, 2n-1, 0) + 2\left(\sum_{i=1}^{2n-1}(P(2n, 2n-1, i-1) + P(2n, 2n-1, i))\right) \\ &+ \sum_{i=1}^{2n-1}\sum_{j=1}^{2n-1} P(2n+1, 2n-1, i, j)\Bigg)\end{aligned}$$

Continuing in this vein would suggest we try to find explicit formulas for the number of tilings of aztec rectangles. This is an unnecessary and extremely complicated plan for reducing this formula, the ‘really hard way’. Instead, we need to come up with a more intelligent way of eliminating the reference to aztec rectangles (proceeding along the ‘hard way’). We can do this by employing a few simple combinatorial arguments. By symmetry, P(2n,2n-1,0)=P(2n,2n-1,2n-1), and so after splitting the single sum into two sums, adding and subtracting twice P(2n, 2n-1, 0), moving this inside each of the sums, and then recombining, we get:

$$\begin{aligned}N(4n) &= 2^{1-n}\Bigg(4\left(\sum_{i=0}^{2n-1} P(2n, 2n-1, i)\right) - 2P(2n, 2n-1, 0) \\ &+ \sum_{i=1}^{2n-1}\sum_{j=1}^{2n-1} P(2n+1, 2n-1, i, j)\Bigg)\end{aligned}$$

Now, as we noted before P(2n, 2n-1, 0) = AZ(2n-1). Also, now that we have extended the index on the single sum down to 0, we note that this sum counts precisely half of the matchings of an aztec diamond of order 2n. (To see this, consider the 2n vertices on the line running northeast-southwest, one block from the lower right side of the aztec diamond of order 2n:

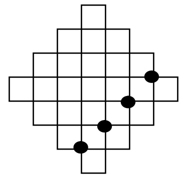


**FIGURE 11. An Order 4 Aztec Diamond with Vertices for Aztec Rectangle Reduction**

Note that precisely one of these will be matched to the south or the east, and for each vertex, matched either south or east, the remaining region is an aztec rectangle.) Thus we have:

$$2\sum_{i=0}^{2n-1} P(2n, 2n-1, i) = AZ(2n)$$

We now retrace our steps slightly and realize that if we can enumerate the number of tilings of modified aztec diamonds, then we need only determine two of the three cases. We use this fact to write the double sum as the number of tilings of a Modified Aztec Diamond (defined at the beginning of 3.3)less the number of tilings in cases one and two. Employing similar combinatorial arguments as those which led to the formula for the single sum above, we arrive at:

$$\begin{aligned} 4\sum_{i=1}^{2n-1}\sum_{j=1}^{2n-1} P(2n+1, 2n-1, i, j) &= MAZ(2n+1) - 2P(2n, 2n-1, 0) \\ &\quad - 4\sum_{i=1}^{2n-1}(P(2n, 2n-1, i-1) + P(2n, 2n-1, i)) \\ &= MAZ(2n+1) - \left(8\left(\sum_{i=0}^{2n-1} P(2n, 2n-1, i)\right) - 6P(2n, 2n-1, 0)\right) \\ &= MAZ(2n+1) - 4AZ(2n) + 6AZ(2n-1) \end{aligned}$$

Substituting these two results into the formula yields:

$$N(4n) = 2^{1-n}(2^{-2}MAZ(2n+1) + AZ(2n) - 2^{-1}AZ(2n-1))$$

We have now almost finished the reduction. We need only determine a formula for the number of tilings of an unweighted modified aztec diamond. The number of tilings of a modified aztec diamond of order n is just the number of tilings of an aztec diamond of order n minus those tilings where a corner edge is not matched. If we match any one of these corner vertices elsewhere from its partner corner vertex, everything except an aztec diamond of order n-1 is forced. In the order three case, we are matching the circled edge, and the region which remains is outlined:

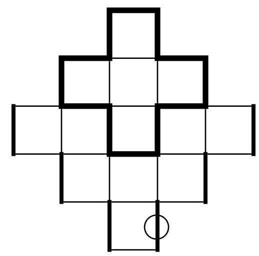


**FIGURE 12. An Order 3 Aztec Diamond Showing Forcing when Corner Dominos are Not Matched**

There are four corners, and thus four ways of doing this. However, we have double counted those matchings where two of the four corners (which must be opposite one another) are not matched. In each of the two cases, everything is forced except an aztec diamond, this time of order n-2. Thus:

$$MAZ(n) = AZ(n) - 4AZ(n-1) + 2AZ(n-2)$$

Substituting this into the formula, along with

$$AZ(n) = 2^{\left(\frac{n(n+1)}{2}\right)}$$

from [4], yields:

$$N(4n) = 2^{1-n}(2^{-2}AZ(2n+1)) = 2^{2n(n+1)},$$

precisely as conjectured.

### 3.4 Cleaning up the Argument (or "Why didn't I see that the first time around?!")

Unfortunately, when you are developing the proof of a conjecture, it doesn't always come to light in its simplest, shortest, or most logical fashion the first time. Because of this, proofs can often be significantly streamlined and improved upon reflection. For example, the following observation, due to Henry Cohn [2], simplifies the case analysis in this proof. Let us return to the n=2 case in the following form:

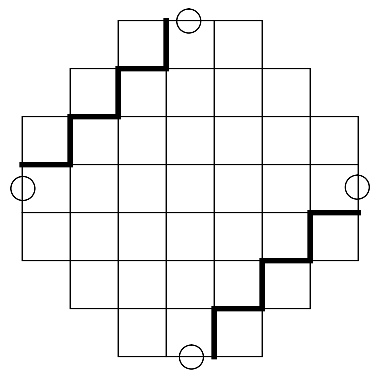


**FIGURE 13. The Order Eight Graph with Edges Marked for Case Division**

Before we argued that at most one half weighted edge on each side could be matched, and so there were three cases. However, we could equally well have proceeded by noting that at most one of the circled edges can be matched on each side, and so we can separate three cases according to how many of these edges are matched. If two are matched, then as before:

$$N_0 = 2(AZ(2n-1))$$

The main simplification comes in considering one matching, as here, upon filling in one circled edge, everything is forced except an aztec diamond with a single pair of corner vertices matched. Now some of these matchings contain two circled edges, but we already know the number of these, which we can subtract off from the number for the aztec diamond with a corner domino inserted (which we effectively worked out along with the modified aztec diamonds at the end of 3.3). The third case is handled similarly to before by writing the number of tilings as the number of matchings of the modified aztec diamond less cases one and two. Combining this idea with the primary computations of the original proof yields a smoother proof which avoids the references to aztec rectangles.

## 4.0 Conclusion

The techniques presented here provide a reasonably efficient method for exploring whether an enumerative conjecture for a tiling region can be reduced to the case of weighted aztec diamonds, and if it can, carrying out this reduction. However, this process is still somewhat unsystematic, and a more refined method for producing conjectures which will definitely reduce to previously known results would be very useful. Toward this end, Henry Cohn proposed the idea of 'working backwards' [2], beginning with an aztec diamond or lozenge tiling, performing, in reverse, the graph manipulation techniques such as urban renewal and vertex splitting, and finally adjusting the weightings on the original graph to accommodate the final structure. In aztec diamonds, for example, by reverse urban renewing different patterns of squares from the original graph, we my be able to actively produce various natural tiling patterns which we know a priori will be easily reduced to weighted aztec diamonds, yielding new enumerative formulas.

## 5.0 References

1. Federico Ardila, Term Paper, Spring 1996

2. Henry Cohn, e-mail to tiling@math.mit.edu, dated April 24, 1996

3. Chris Douglas, e-mail to tiling@math.mit.edu, dated April 23, 1996

4. Noam Elkies, Gregory Kuperberg, Michael Larsen, and James Propp, *Alternating Sign Matrices and Domino Tilings*, J. of Alg. Comb. **1** (1992), 111-132 and 219-234

5. James Propp, e-mail to domino@math.mit.edu, dated February 22, 1996