A Connection Between Perfect Matchings and Spanning Trees of Planar Graphs

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Perfect Matchings and Spanning Trees

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Abstract: Given a finite connected graph G on the sphere, let $\mathcal{H}(G)$ be the Hasse diagram of the face-poset of G, viewed as a graph. I show that if one deletes from $\mathcal{H}(G)$ two nodes corresponding to a vertex of G and a face of G incident to that vertex, then the perfect matchings of the resulting graph are in one-one correspondence with the spanning trees of G. This generalizes Temperley's result on spanning trees in a square grid.

List of special symbols: $\mathcal{H}$

Let G be a finite connected graph embedded in S^2, with multiple edges, self-loops, pendant edges, and isthmuses allowed. Put V = the set of vertices of G, E = the set of edges, F = the set of faces. Define $\mathcal{H}(G)$ as the graph with a node $\overline{v}$ corresponding to each vertex v of G, a node $\overline{e}$ corresponding to each edge e of G, and a node $\overline{f}$ corresponding to each face f of G, with an edge joining two nodes in $\mathcal{H}(G)$ if the corresponding structures in G are either an edge and one of its endpoints *or* an edge and one of the faces it bounds. $\mathcal{H}(G)$ is the Hasse diagram of the face-poset of G. Pictorially, we may derive $\mathcal{H}(G)$ from G by adding a new node on each edge e and a new node on each face f and joining them by a new edge if e is part of the boundary of f. For instance, if G is the graph shown in on the left in Figure 1, $\mathcal{H}(G)$ is the graph shown on the right.

Let v^* be a vertex of G and f^* a face of G, and let $H = H(v^*, f^*)$ be the graph $\mathcal{H}(G)$ with the nodes $\overline{v^*}, \overline{f^*}$ deleted (along with all incident edges in $\mathcal{H}(G)$). Since by Euler's formula $(\#(V) - 1) + (\#(F) - 1) = \#(E)$, H is a balanced bipartite graph.

THEOREM: If v^* is incident with f^*, then the number of perfect matchings of $H(v^*, f^*)$ is equal to the number of spanning trees of G. If v^* is not incident with f^*, then the number of perfect matchings of $H(v^*, f^*)$ is greater than or equal to the number of spanning trees of G.

Note that in particular, the first part of the theorem implies that the number of perfect matchings of $H(v^*, f^*)$ is independent of v^* and f^*, provided that v^* and f^* are incident with one another. Thus there are as many as $2\#(E)$ distinct graphs (one for each choice of an incident pair v^*, f^*) that are guaranteed to have the same number of perfect matchings.

This theorem, along with its proof, is a generalization of a result of Temperley that originally appeared in [3] and is discussed in problem 4.30 of [2]. The result was independently discovered by Burton and Pemantle, who apply it to infinite graphs [1].

Henceforth, we refer to perfect matchings as simply "matchings."

PROOF OF THEOREM: It will be enough to exhibit an injective mapping from the set of spanning trees of G into the set of matchings of H, and to show that when v^* is incident with f^*, every matching of H arises from a spanning tree of G.

Let $G^\perp$ be the dual graph of G, with vertices, edges, and faces of $G^\perp$ corresponding to faces, edges, and vertices of G, respectively. We can embed G and $G^\perp$ simultaneously in S^2, such that an edge e of G crosses the

corresponding dual edge $e^\perp$ of $G^\perp$ exactly once and crosses no other edge of $G^\perp$. In fact, the joint embedding of G and $G^\perp$ is another way of constructing $\mathcal{H}(G)$.

Given a spanning tree T of G, the set of edges of $G^\perp$ that do not cross edges of T is a spanning tree of $G^\perp$, called the dual tree and here denoted by $T^\perp$. Orient the edges of T so that they point towards v^* (i.e., orient edge v_1v_2 towards v_2 iff v_2 lies between v_1 and v^*), and orient the edges of $T^\perp$ so that they point towards f^*. Then a matching M of H can be obtained as shown in Figure 2. Specifically, for each $v \in V$, pair $\overline{v}$ with the unique $\overline{e}$ such that v is an endpoint of e and e is pointing away from v in the orientation of T, and for each $f \in F$, pair $\overline{f}$ with the unique $\overline{e}$ such that e bounds f and $e^\perp$ is pointing away from f in the orientation of $T^\perp$.

To verify that this gives a matching M of H, it suffices to show that no edge-node $\overline{e}$ is paired twice. But this could only happen if we had $e \in T$ and $e^\perp \in T^\perp$, contradicting the definition of a dual tree.

From the matching M we can easily recover T as the set of edges e such that $\overline{e}$ is paired with a vertex-node in H under the matching M. Hence the mapping $T \mapsto M$ is injective.

Now suppose v^* is incident with f^*, and let M be a matching of H. Let $\widetilde{T}$ be the set of edges e of G such that $\overline{e}$ is paired with a vertex-node by M. To complete the proof of the theorem, we must show that $\widetilde{T}$ is a spanning tree. Note that $\widetilde{T}$ spans G and has $\#(V) - 1$ edges, so it suffices to prove that $\widetilde{T}$ is acyclic.

Suppose $\widetilde{T}$ contained a cycle C, say of length n. C divides the sphere into two (open) parts, one of which contains both v^* and f^* and the other of which contains neither. I claim that each hemisphere contains an odd number of nodes of $\mathcal{H}(G)$ and hence an odd number of nodes of H as well. For, suppose we modify G by replacing either of the two hemispheres by a single face. By Euler's formula, the number of vertices, edges, and faces in the resulting graph must be even. Since there are an even number of these elements on the cycle C (n vertices and n edges) and an odd number in the modified hemisphere (1 face), the unmodified hemisphere must have an odd number of elements as well.

Since the edges of C disconnect H into parts lying in the two hemispheres, M must match each hemisphere within itself. But this is impossible, since each hemisphere has been shown to contain an odd number of nodes of H. This completes the proof of the theorem. □

As was remarked earlier, the theorem implies that when v' is incident with f' and v'' is incident with f'', the matchings M' of $\mathcal{H}(v', f')$ are equinumerous with the matchings M'' of $\mathcal{H}(v'', f'')$; in fact, the proof of the theorem provides a bijection between the two sets of matchings. This bijection can be understood without reference to spanning trees, as a process of "sliding edges." Specifically, one finds a chain

$$v'' = v_0,\ e_0,\ v_1,\ e_1,\ v_2, ..., e_{r-1},\ v_r = v'$$

such that the endpoints of e_i are v_i and v_{i+1}, and such that M' pairs $\overline{e_i}$ with $\overline{v_i}$ $(0 \leq i < r)$, and modifies the matching by pairing $\overline{e_i}$ with $\overline{v_{i+1}}$ instead. One then does the same with a chain of dual-edges joining the faces f'' and f'.

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References

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