The Game of Cutblock and Surreal Vectors

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Conway's theory of partizan games is at once a theory of *games* and a theory of *numbers.* I hope to convince you that there is a good chance that we will be able to generalize the theory to more than two players while retaining its dual nature. More good ideas are needed, however, before the theory can get off the ground.

1. Games for three players.

In the game of Cutblock (a three-player variant of Cutcake), a position is a collection of blocks of integer side-lengths, with edges parallel to the x-, y-, and z-axes. Let us call the players X, Y, and Z. A legal move for X is to divide one of the blocks into two blocks of integer side-lengths by means of a single cut perpendicular to the x-axis; legal moves for Y and Z are defined similarly. Players take turns making legal moves in cyclic fashion $(..., X, Y, Z, X, Y, Z, ...)$ until one of the players is unable to move. Then that player leaves the game and the other two continue in alternation until one of them cannot move. Then that player leaves the game, and the remaining player is deemed the winner.

In general, none of the three players has a winning strategy; that is, it is typically the case that any two players, acting in concert throughout the game (with one of them possibly acting in disregard of his or her own best interest), can prevent the third player from winning. For instance, let $[a, b, c]$ denote an a-by-b-by-c block, and let addition stand for the operation of laying several blocks together to form a single position. Then $[1, 2, 2] + [2, 1, 2] + [2, 2, 1]$ has the property that any two players can gang up on the third.

Still, there are many positions in which one of the players does have a surefire way to win. In particular, positions of the form $[a, b, c]$ seem to have this property when a, b, c are small. Indeed, when $a, b, c \leq 3$, the identity of the player with the winning strategy in the game with initial position $[a, b, c]$ is given by the following table:

$a = 1$:

	$b = 1$	$b = 2$	$b = 3$
$c = 1$	XYZ	Y	Y
$c = 2$	Z	YZ	Y
$c = 3$	Z	Z	YZ

$a = 2$:

	$b = 1$	$b = 2$	$b = 3$
$c = 1$	X	XY	Y
$c = 2$	XZ	XYZ	Y
$c = 3$	Z	Z	YZ

$a = 3$:

	$b = 1$	$b = 2$	$b = 3$
$c = 1$	X	X	XY
$c = 2$	X	X	XY
$c = 3$	XZ	YZ	XYZ

If a box in this table is marked with more than one letter, then the player with the winning strategy is the one (among those listed) who makes his or her first move *last*, under cyclic play.

I conjecture that $[n, n, n]$ is always a win for the third player.

Also, recall that in the two-player theory, the a-by-b rectangle is a win for X, a win for the second player, or a win for Y, according to whether $A > B$, $A = B$, or $A < B$, where $A = 2^{\lceil \log_2 a \rceil}$ and $B = 2^{\lceil \log_2 b \rceil}$. If one analogously defines $C = 2^{\lceil \log_2 c \rceil}$, then one might conjecture that the winner of $[a, b, c]$ is determined by whichever of A, B, C is largest, with tied comparisons being decided by which of the tied players makes his or her first move last. (For instance, if $[a, b, c] = [4, 5, 6]$, then $A = 4$ and $B = C = 8$, so the player with the winning strategy is whichever one of Y, Z makes his or her first move last.)

This conjecture is supported by data for $1 \leq a, b, c \leq 4$, $a + b + c \leq 10$.

2. Numbers for the plane.

Let L be a triangular lattice of points in the plane, as shown in Figure 1, with an origin called 0. The lattice point immediately above 0 will be called 1; reading clockwise, the six neighbors of 0 will be called 1, $-\zeta$, ζ^2, -1, ζ, $-\zeta^2$. (You may wish to view these points as complex numbers, in a coordinate system that has been rotated 90 degrees relative to the usual orientation.) Points in the lattice correspond to vectors in the plane and thus may be added to one another.

Figure 1

The rays joining 0 to the points $-\zeta$, -1, and $-\zeta^2$ divide the plane into three (open sectors), which we will call north, southeast, and southwest in the obvious (albeit cartographically inaccurate) way. We say that one point p is **north** of another point q iff $p-q$ is in the northern sector. Similarly for southeast and southwest.

Every lattice point in the closure of the northern sector can be written uniquely as $-t\zeta - u\zeta^2$, with $t, u \geq 0$. We define the **simplicity** of this point as $t+u$. The simplicity of a point in the closure of a different sector is defined analogously. Figure 2 shows the simplicities of various points near the origin. We say that one point is **numerically simpler** than another if its simplicity is numerically smaller.

Figure 2

Claim 1: Fix subsets P, Q, and R of L. If there exists a point that is southeast of everything in P, north of everything in Q, and southwest of everything in R, then there is a unique numerically simplest point with this property, which we denote by or $\{P/Q\backslash R\}$ for short. When P, Q and R are specific sets, written in set-builder notation, we can omit one layer of brackets; e.g., $\{a,b/\ \backslash c\}$ is short for $\{\{a,b\}/\{\}\backslash\{c\}\}$.

Claim 2: If $o=\{P/Q\backslash R\}$ and $o'=\{P'/Q'\backslash R'\}$, then

$$o+o'=\{\,o+P',\,o'+P\,/\,o+Q',\,o'+Q\,\backslash\,o+R',\,o'+R\,\}\,,$$

where $o+P'=\{o+p':p'\in P'\}$, etc.

Both claims are easily proved, and are analogous to results that apply for two-player games.

3. Numbers for games' sake.

We can recursively define the **value** of a Cutblock position as $\{P/Q\backslash R\}$, where P (resp. Q, R) is the set of values of positions reachable from the given position by player X (resp. Y, Z). In calculating this, it is helpful to prove and use a lemma which says that the value of a sum of positions is the sum of the values of those positions.

If one computes the values of positions $[a,b,c]$ with $1\le a,b,c\le 4$, one finds results consistent with the following conjecture: the value of $[a,b,c]$ is given by $s+t\zeta+u\zeta^2$, where $2^{s-1}<a\le 2^2$, $2^{t-1}<b\le 2^t$, and $2^{u-1}<c\le 2^u$. Note that this accords with the game-theoretic analysis, in the weak sense that games that are last-player-wins have value 0.

Of course, the values of positions are not as informative here as they were in the two-player theory. It is not the case that two games with the same value are equivalent; for instance $[1,1,2]+[1,2,1]+[2,1,1]$ has a winning strategy for the third player while $[1,2,2]+[2,1,2]+[2,2,1]$ has a winning strategy for none of the players — this despite the fact that both positions have value 0. Still, I believe that this lattice of values forms a scaffolding on which to hang the theory of Cutblock, just as the surreal numbers form a useful framework for the analysis of two-player games that are not themselves surreal numbers but are approximable by them. For instance, proving that $[n,n,n]$ is always a win for the third player will probably require estimates of which moves strategically dominate other moves; L would be a natural setting in which such comparisons could take place.

If Cutblock yields to this sort of analysis, then other cold two-player games, extended to include a third player, might be approached in similar fashion.

4. Numbers for numbers' sake.

The Conway way of constructing numbers gives us far more than just $\mathbf{Z}$, so we would expect our planar numbers to yield more general "surreal vectors." For instance, working inside $\mathbf{R}^2$ with our concrete picture of L as a subset, we can recognize that $\{0/\zeta^2 \backslash -\zeta\}$ "ought" to have some value in $\mathbf{R}^2$; but what particular value?

Rather than trying to enlarge L inside a pre-existing copy of $\mathbf{R}^2$, we could try to proceed "intrinsically": if P, Q, and R are sets of already-constructed "surreal vectors," then $\{P/Q\backslash R\}$ should be taken as defining a new surreal vector, provided certain conditions are satisfied. These conditions will involve certain relations between P, Q, and R which must themselves be given recursive definitions. Finding the right rules could be difficult, but once it's been done, the formulation of autonomous rules for describing the properties of constructed objects from the properties of their constituents opens up the possibility of taking the whole theory to an interesting infinite limit.

Leaving aside these transfinite ambitions, we might content ourselves more modestly with a good Conway-style definition of multiplication in L, consistent with the way complex numbers in $\mathbf{Z}[(1+\sqrt{-3})/2]$ multiply. I think it's a blemish, esthetically speaking, that the surcomplex numbers have been defined as the surreal numbers with $\sqrt{-1}$ adjoined; this trick is not in keeping with the spirit of Conway's construction of the surreals. It would be wonderful if the theory of multi-player games gave us a natural way of arriving at the complex numbers without recourse to algebraic adjunction of formal indeterminates, which is an effective but lackluster device.