Number Walls in Combinatorics

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September 20, 2000

A Catalan Number Wall

1	1	1	1	1	1	1	1
1	*1*	2	5	14	42	132	429
-1	*1*	*1*	3	14	84	594	4719
-1	-2	*1*	*1*	4	30	330	4719
1	-5	-3	*1*	*1*	5	55	1001
1	14	-14	-4	*1*	*1*	6	91
-1	42	84	-30	-5	*1*	*1*	7
-1	-132	594	330	-55	-6	*1*	*1*
1	-429	-4719	4719	1001	-91	-7	*1*

A Somos-4 Number Wall

1	1	1	1	1	1	1	1
1	1	3	8	23	68	207	644
1	*2*	-1	5	15	137	943	7544
-1	-3	*3*	-5	20	-68	697	2050
1	-6	-6	*7*	-4	68	-663	8627
1	14	-26	5	*23*	29	211	-1414
-1	37	101	-89	96	*59*	129	1405
-1	-105	519	355	-629	307	*314*	-65
1	-312	-3036	5084	-2986	-4945	-919	*1529*

A Ternary Tree Number Wall

1	1	1	1	1	1	1	1
1	0	1	0	3	0	12	0
-1	*1*	-1	3	-9	36	-144	660
-1	0	*2*	0	9	0	252	0
1	-2	4	*6*	9	63	441	5271
1	0	-14	0	*33*	0	546	0
-1	7	-49	-77	-121	*286*	-676	10036
-1	0	210	0	-1111	0	*4420*	0
1	-30	900	3030	10201	-17170	28900	*109820*

A Ternary Tree Number Wall

1	1	1	1	1	1	1
1	*1*	3	12	55	273	1428
-1	*2*	*3*	21	251	4011	77112
-1	-7	*11*	*26*	386	11967	571797
1	-30	-101	*170*	*646*	19323	1427994
1	143	-1391	-3621	*7429*	*45885*	2677545
-1	728	24284	-137914	-342631	*920460*	*9304650*

The fundamental equation is

$$x = y - y^2 = z - z^3.$$

First, expand y as a series in x giving

$$y = 1x + 1x^2 + 2x^3 + 5x^4 + 14x^5 + 42x^6 + \ldots$$

the generating function for Catalan numbers. Its number wall has a zigzag diagonal of all ones.

Now expand y as a series in z giving

$$y = 1z + 1z^2 + 1z^3 + 3z^4 + 8z^5 + 23x^6 + \ldots$$

the generating function of a sequence related to Catalan and Motzkin numbers. Its number wall has a Somos-4 sequence diagonal. See sequence A006769 in Sloane's EIS for the zigzag diagonal.

Next expand z as a series in x giving

$$z = 1x + 1x^3 + 3x^5 + 12x^7 + 55x^9 + \ldots$$

the generating function for ternary trees. Its number wall including zero coefficients has one diagonal sequence A005161 enumerating a symmetry class of alternating sign matrices invariant under vertical and horizontal reflections.

Furthermore, its number wall without zero coefficients has one diagonal sequence A051255 enumerating a symmetry class of planar partitions and another diagonal sequence A005156 enumerating a symmetry class of alternating sign matrices invariant under vertical reflection.

See my `grail.cba.csuohio.edu/~somos/` or `www.research.att.com/~njas/sequences/`