REACH meeting, February 19, 2002, 4-5:30pm

• Next Monday: 10-15 minute talks with Jim and Jason; Jim will post times and places.
• Jim will post previous minutes on Web.
• Gregg reports:

Let $f(0) = f(1) = g(0) = g(1) = x$, $f(n)f(n-2) = f(n-1)^2 + z^2$, $g(n)g(n-2) = g(n-1)^2 + z^n$. Then $f(2) = g(2) = (x^2+z^2)/x$, $\text{num}[f(3)] = x^4+3z^2x^2+z^4$. $f(n)$ counts the weighted perfect matchings of the dual graph to the 2-by-$2n$ rectangle with vertical edges weighted x and horizontal edges weighted z

Also, $\text{num}[g(3)] = x^4 + 2x^2z^2 + z^4 + z^3x^2$, and as it turns out $g(n)$ counts the weighted perfect matchings of the dual graph to the 2-by-$2n$ rectangle with vertical edges weighted x, the leftmost, third-from-leftmost, fifth-from-leftmost, etc. pairs of horizontal edges weighted z, and the other horizontal edges weighted as some function of z.

• Jim gives a 2-D version of this recurrence:

First we define, slightly more generally, $f(0) = x, f(1) = y, f(n)f(n-2) = f(n-1)^2+z^n$. This can be extended to the 2-D recurrence $f(n,k)f(n-2,k) = f(n-1,k-1)f(n-1,k+1) + z^n$. With arbitrary initial conditions, we get:

x_0 x_2 x_4 x_6
y_1 y_3 y_5
A B
C

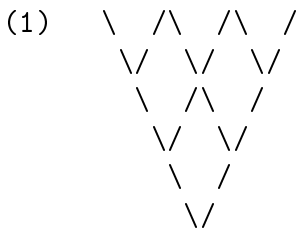
where $A = \frac{y_1y_3+z^2}{x_2}$, $B = \frac{y_3y_5+z^2}{x_4}$, $C = AB + z^3y^3$, etc.

Claim: all terms are Laurent series with unit coefficients.

• Gregg reports:

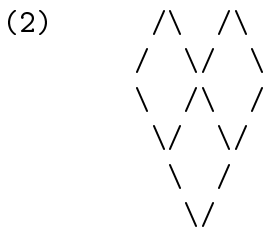
There are bijections between the sums of the weighted perfect matchings of the following three graphs:

(1)

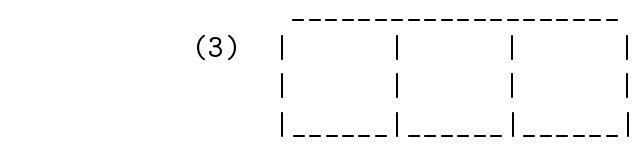


with each vertex in the top row weighted x, each vertex in the second row weighted y/x^2, each vertex in the third row weighted x/y^2, and the vertex in the last row weighted y.

(2)



with the *edges* in the top row weighted $x, y/x^2, x, y/x^2$ from left to right, the edges in the middle row weighted $x/y^2, y/x^2, x/y^2, y/x^2$ from left to right, and the edges in the bottom row weighted $y, x/y^2$ from left to right.



with the leftmost and rightmost upper horizontal edges weighted y/x^2, the middle upper horizontal edge weighted x/y^2, the leftmost and rightmost lower horiztonal edges weighted x, the middle lower horizontal edge weighted y, and the vertical edges weighted x/y^2, y/x^2, x/y^2, y/x^2 from left to right; and such that the weighting of a perfect matching is multiplied by $1/y$.

• Homework #2 (to be discussed 2/26/02 in class): Let $P(a, b)$ be the "lotsa variables polynomial" for perfect matchings of the grid with vertex set $\{a, a+1, \ldots, b\} \times \{0, 1\}$. Put an edge between (i, j) and (k, l) iff $|k - i| + |j - l| = 1$. Weight the edges as follows: the edge between $(i, 0)$ and $(i, 1)$ has weight z_i, the edge between $(i-1, 1)$ and $(i, 1)$ has weight x_i, and the edge between $(i-1, 0)$ and $(i, 0)$ has weight y_i. For instance, $P(2, 2) = z_2$, $P(1, 2) = x_2y_2 + z_1z_2$, $P(2, 4) = z_2z_3z_4 + z_2x_4y_4 + x_3y_3z_4$. Come up with a simple formula for the quantity

$$P(2a, 2b)P(2a-2, 2b+2 - P(2a-2, 2b)P(2a, 2b+2), \qquad a \leq b.$$

• David reports, similarly to as in Kuo's paper:

The dual graph of the Aztec diamond is defined as follows: to construct this graph, put a dot in the middle of each unit box of the diamond and connect two dots iff their corresponding squares share a common edge. Perfect matchings of this graph are in bijection with domino tilings of the Aztec diamond.

The number of tilings of the order-n Aztec diamond is $T_n = 2^{n(n+1)/2}$, which can be proven by the recurrence $2T_n^2 = T_{n+1}T_{n-1}$.

David then highlights the arguments of the paper.

Another interesting object of study is the 3-D object constructed as follows: construct a lattice plane. Put a lattice plane with the same basis vector above it so that, when looking from above, each of its points lies in the middle of a square formed by four neighboring lattice points in the first lattice. Continue this process.

Label four neighboring vertices in any one plane B, C, D, E, clockwise. Label the vertex in the middle of these immediately above them A and the vertex in the middle of these immediately below them F. Then one can study the sequences formed by the recurrence $AF = BE + CD$ (or $AF = BE - CD$).

• Jim now returns to a slightly-simplified previous example:

$$\begin{array}{ccccccc} x_0 & & x_2 & & x_4 & & x_6 \\ & y_1 & & y_3 & & y_5 & \\ & & \frac{y_1y_3-1}{x_2} & & \frac{y_3y_5-1}{x_4} & & \end{array}$$

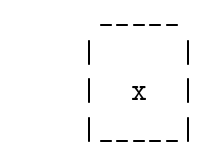
One can make this more complicated in two ways. First, by increasing the complexity of the initial two rows. Second, by increasing the complexity of the recurrence, as in:

$$\begin{array}{ccccccccc} 1 & & 1 & & 1 & & 1 & & 1 \\ & 1 & & 1 & & 1 & & 1 & \\ & & A & & B & & C & & \\ & & & D & & E & & & \\ & & & & F & & & & \end{array}$$

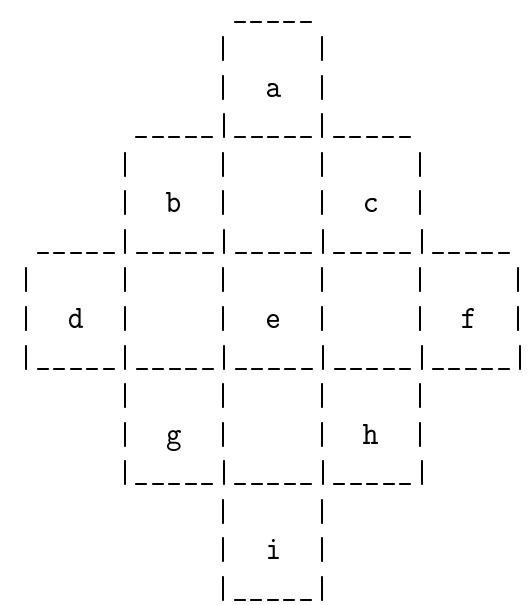
where $A = \frac{1\cdot 1+w_0}{1}$, $B = \frac{1\cdot 1+w_2}{1}$, $C = \frac{1\cdot 1+w_4}{1}$, $D = \frac{1\cdot 1+w_1'}{1}$, $E = \frac{1\cdot 1+w_3'}{1}$, $F = \frac{DE+w_2''}{B}$.

• Jim discusses a weighted dual graph of the Aztec diamond.

For ease of notation, denote the square graph whose upper vertical, leftward horizontal, rightward horizontal, and lower vertical edges are weighted x_1, x_2, x_3, x_4 by



Further, if every edge of a graph is weighted, we need only label every other box, so for instance the weighted graph



can be represented uniquely as

$$\begin{array}{ccccc} & & \mathrm{a} & & \\ & \mathrm{b} & & \mathrm{c} & \\ \mathrm{d} & & \mathrm{e} & & \mathrm{f}\ . \\ & \mathrm{g} & & \mathrm{h} & \\ & & \mathrm{i} & & \end{array}$$

Let $P(G)$ be the sum of the weights of the perfect matchings of the graph G. Then

$$P\left(\begin{array}{ccccc} & & \mathrm{a} & & \\ & \mathrm{b} & & \mathrm{c} & \\ \mathrm{d} & & \mathrm{e} & & \mathrm{f} \\ & \mathrm{g} & & \mathrm{h} & \\ & & \mathrm{i} & & \end{array}\right) P(e) = d_2 f_3 P\left(\begin{array}{ccc} & \mathrm{a} & \\ \mathrm{b} & & \mathrm{c} \\ & \mathrm{e} & \end{array}\right) P\left(\begin{array}{ccc} & \mathrm{e} & \\ \mathrm{g} & & \mathrm{h} \\ & \mathrm{i} & \end{array}\right) + a_1 i_4 P\left(\begin{array}{ccc} & \mathrm{b} & \\ \mathrm{d} & & \mathrm{g} \\ & \mathrm{e} & \end{array}\right) P\left(\begin{array}{ccc} & \mathrm{c} & \\ \mathrm{e} & & \mathrm{f} \\ & \mathrm{h} & \end{array}\right)$$

just as

$$AF = BE + CD.$$

See

`http://www.math.wisc.edu/~propp/somos/domino`

and

`http://www.math.wisc.edu/~propp/somos/kuo3`

for more accounts of this stuff.