REACH meeting, October 23, 2001, 4-5:30pm

- Submit monthly problems to:
  Doug Hensley, Monthly Problems
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  Texas A & M University
  College Station, TX 77840

Send two copies, typed on one side only. Mention REACH

- Software: label emails "bowtie," "Deutsch," etc. ... type as few symbols as possible as a rule of thumb

Jim said he would put a new version of `graph.tcl` on the Webpage, post a link to the `wish` interpreter, post `planemaple.tcl`, and make a strip tiling program (that calculates generating functions for tilings of arbitrary rectangles) available

Matt will port planemaple into planemathematica

- Why should counting negative elements multiple times work? (1) Polytopes and Euler characteristic. (2) Generating function approach: consider the series 2^n run backwards. Associate variables with each element as follows:

$$8 \quad (1+a_1t)(1+a_2t)(1+a_3t) = \sum_{S\subseteq\{1,2,3\}} \left(\prod_{i\in S} a_i\right) t^{|S|} = \sum_k \left(\sum_{S\subseteq\{1,2,3\}:|S|=k} \left(\prod_{i\in S} a_i\right)\right) t^k$$
$$= 1 + (a_1+a_2+a_3)t + (a_1a_2+a_1a_3+a_2a_3)t^2 + (a_1a_2a_3)t^3$$

$4 \quad (1+a_1t)(1+a_2t)$

$2 \quad (1+a_1t)$

$1 \quad 1$

$\frac{1}{2} \quad (1+a_0t)^{-1} = 1 - a_0t + a_0^2t^2 - a_0^3t^3 + \cdots$

$\frac{1}{4} \quad (1+a_0t)^{-1}(1+a_{-1}t)^{-1} = 1 - (a_0+a_{-1})t + (a_0^2 + a_0a_{-1} + a_{-1}^2)t^2 + \cdots$

Thus algebra can be said to "force" the interpretation that one can multichoose negative elements. Consider setting $a_i = 1$, looking at the expansions as power series, to get $\frac{1}{2}$, $\frac{1}{4}$, etc.

- In a 2-by-n rectangle (divided up into $2n$ squares), label the middle vertical edges formed by the union of the squares $z_1, z_2, z_3, \ldots$, the upper horizontal edges $x_0, x_1, x_2, \ldots$, and the lower horizontal edges $y_0, y_1, y_2, \ldots$. Identify the weight of a domino tiling of a rectangle by a product of the edges that are selected (if an edge is selected, the two squares that overlap it form a domino). Then we can write the sum of the weights of the tilings of the rectangle as follows, running backwards:

$x_1y_1z_3 + x_2y_2z_1 + z_1z_2z_3$

$x_1y_1 + z_1z_2$

z_1

1

0

$\frac{1}{x_{-1}y_{-1}}$

$-\frac{z_{-1}}{x_{-1}x_{-2}y_{-1}y_{-2}} = -x_{-1}^{-1}x_{-2}^{-1}y_{-1}^{-1}y_{-2}^{-1}z_{-1}^{+1} - \cdots$, a Laurent polynomial

$\frac{x_{-2}y_{-2}+z_{-1}z_{-2}}{x_{-1}x_{-2}x_{-3}y_{-1}y_{-2}y_{-3}}$

Notice the numerators are essentially the same for the nth and $-n$th term in the sequence, up to a minus sign. How do we bring the denominators into the picture? Open problem.

• Bijection between (1) the domino tilings of a 2-by-$2n$ rectangle and (2) the tilings of a 2-by-$3n$ rectangle with (i) 2-by-2 squares with the lower-right corner missing, (ii) 2-by-2 squares with the upper-left corner missing, and (iii) tiles of width 3 and length 1 (get every other Fibonacci number). The bijection can be seen by removing one square from each of the blocks in the second case (turning (i) and (ii) into vertical dominoes and (iii) into horizontal dominos). In case (1), the tiling can include blocks of two dominoes stacked on top of each other or two single dominos separated by any number of such blocks. In case (2), the tiling can include two blocks of type (iii) stacked directly on top of each other or a block of type (i) followed by pairs of offset blocks of type (iii) stacked on top of each other followed by a block of type (ii).

• Popoviciu and Convergence:

$\frac{1}{1-x} = 1 + x + x^2 + \cdots = \sum\limits_{n\geq 0} x^n \quad \text{if } |x| < 1$

$-\frac{1}{1-x} = \frac{x^{-1}}{1-x^{-1}} = x^{-1} + x^{-2} + x^{-3} + \cdots = \sum\limits_{n<0} x^n \quad \text{if } |x| > 1$

But we cannot say that $\sum\limits_{n=-\infty}^{+\infty} x^n$ since the above sums have disjoint domains of validity

In general if $f(n)$ satisfies an LRE then $\sum\limits_{n=0}^{\infty} f(n)x^n$, $\sum\limits_{n=-\infty}^{-1} f(n)x^n$ cannot converge simultaneously

• Domino posting from morning of 10/23 on reciprocity relations. Let + and − represent positive and negative numbers, respectively, and T denote a function with a recurrence relation that can be run backwards. If there is a reciprocity relation between $T(+,+)$ and $T(+,-)$ and a reciprocity relation $T(+,+)$ and $T(-,+)$, do these relations commute so that there are reciprocity relations between $T(+,-)$ and $T(-,-)$ and between $T(-,+)$ and $T(-,-)$? In particular, does this hold for the number $T(k,l,m)$ of domino tilings of a k,l,m,k,l,m semi-regular hexagon, coming from the fact that $T(+,+,-) = (-1)^{?}T(+,+,+)$, where ? is some function of k,l,m?

Now let $T(a,b)$ be the number of domino tilings of an a-by-b rectangle. Find a recurrence that links $T(1,-2)$, $T(2,-2)$, $T(3,-2)$, $\ldots$ and run it backwards to get $T(-,-)$.

• Ethan and Billy will take over the following areas of study relating to lozenge tilings of semiregular hexagons, about which Jim will email details to the list: (1) perfect matchings of honeycomb graphs and (2) lattice paths and routings