A Reciprocity Theorem For Box Product Graphs

T. Kyle Petersen John A. Baldwin Nick Anzalone
Ilya Bronshtein

December 16, 2002

Abstract

Let $M(m,n)$ denote the number of matchings of an m by n grid graph

1 Introduction

A *matching* of a graph $G = (V, E)$ is a subset $E' \subset E$ together with the subset $V' \subset V$ of vertices that are not incident with any edge in E'. And we require that for any vertex $v \in V$, v is incident with at most one edge in E'. A matching is called *perfect* if V' is empty - that is, if all edges in V are incident with exactly one edge of E'. To G, we can associate *weights*, which may be numbers or formal variables, to the edges in E and to the vertices in V. Then, we define the weight of a matching of G to be the product of the weights of the vertices and edges in the matching. And we define the weighted number of matchings of G to be the sum of the weights of all of the matchings of G. Note that if we let all weights be 1, we simply obtain the number of matchings of G. If we consider the number of matchings of a rectangular grid-graph of fixed height m, and varying width n, we obtain an integer sequence, $\{a_n\}$.[1] For example, when $m = 1$, we obtain the Fibonacci sequence:

$$1, 1, 2, 3, 5, 8, \ldots$$

This sequence satisfies a linear recurrence, namely $a_n = a_{n-1} + a_{n-2}$. So we can always obtain the value of a term based on the two terms to the left of it. But similarly, we can obtain the value of a term by the two terms to the right of it, e.g. $3 =$ 8-5, or $a_n = a_{n+2} - a_{n+1}$. In this way we can extend the Fibonacci sequence to the left and get values for a_n when $n < 0$:

$$\ldots -8, 5, -3, 2, -1, 1, 0, 1, 1, 2, 3, 5, 8, \ldots$$

So we see that we now have a doubly infinite integer sequence, and that it is symmetric up to sign, i.e. $a_n = \pm a_{-n-2}$. Some natural questions arise: Does

[1]The rectangular grid-graph is the dual graph for monomer-dimer tilings of an m by n rectangle. That is, we first put a vertex in the center of each 1 by 1 unit of the rectangle. A lone vertex represents a square occupied by a monomer while an edge represents adjacent squares covered by a dimer. Throughout the paper we will discuss the problem in terms of graphs, but occasionally we refer to monomer-dimer tilings for emphasis.

this symmetry mean something? Do the values of a_n for $n < 0$, being integers, count something? If so, can we extend the result to larger values of m?

The answer to all the questions is yes. In [1], Propp considered integer sequences generated by perfect matchings of rectangular grid graphs of fixed height, $N(m,n)$, and how they extend to values for $n < 0$. He came up with a unified combinatorial object and a way of counting signed matchings of these objects that allowed him to show that any half-infinite sequence given by the number of perfect matchings of a fixed-height grid graph does three things:

- It satisfies a linear recurrence relation of finite degree.
- The numbers obtained when going backwards are unique and are always integers.[2]
- The bi-infinite sequences have a special kind of symmetry, or *reciprocity*[3]

We will show similar results for general matchings of a rectangular grid graph, using similar tactics. That is, we will define combinatorial objects and a signed enumeration of these objects that will allow us to show analogous statements of uniqueness, integrality, and reciprocity. However, whereas Propp was able to say

$$N(m,n) = \pm N(m, -n-2),$$

we will have a seemingly weaker statement, that

$$M(m,n) = M^*(m, -n-2).$$

Here $M(m,n)$ denotes the number of general matchings rather than only perfect ones and $M^*(m,n)$ will be appropriately defined, but unfortunately most of the time $|M^*(m,n)| \neq |M(m,n)|$.

More generally than rectangular graphs, we can prove results for graphs of the form $G \boxtimes P_n$. We can show that objects exist to give the number of matchings for negative values of n, i.e. a definition for $G \boxtimes P_{-n}$, implying that the same symmetry will exist for this entire family of graphs. Throughout this paper, we will rely heavily on the example of a rectangular grid graph to illustrate key points (notice it is just $G \boxtimes P_n$ where G is just a column of vertices attached by vertical edges), but the reader should keep in mind that everything can apply to the more general box product.

2 Signed Graphs and Signed Matchings

(PICTURE OF COMPONENTS OF A SIGNED GRAPH)

[2] *A priori* the numbers obtained from running a linear recurrence in reverse need only be rational. If the recurrence is of the form $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$, then to push the sequence backwards we solve for the smallest term to get $a_{n-k} = \frac{a_n - c_1 a_{n-1} - \cdots - c_{k-1} a_{n-k+1}}{c_k}$.

[3] The term reciprocity comes from the generating function approach to the problem. See the section entitled "Reciprocity" for more on this view of things.

We begin by giving a slightly more general definition of a graph. Let $G = (V, E)$ be a graph in the usual sense, except that there is more than one kind of edge and more than one kind of vertex. There are plain, anti-, and empty vertices, plain and anti-vertical edges (vedges), and plain horizontal edges (hedges). For our purposes, plain components are given weight 1, anti-vertices and anti-vedges are given weight -1, and empty vertices have weight zero. We could attach formal variables to edges and vertices, but empty vertices would still have weight zero. More will be said about this later.

Define a *signed matching* of a graph G to be a collection of non-adjacent edges of G and all the vertices not incident with those edges. The weight of the matching is the product of the weights of the components in the matching. Note that this means all the matchings with empty vertices vanish.[4] In practice, it helps to think of empty vertices as being *needy*. For any nonzero matching, empty vertices *need* to have one of their adjacent edges included.

(PICTURE OF THE GRAPHS OF $G(2,2)$ AND OF $G(3,-3)$, MAYBE $G^*(2,2)$)

Now we define a regular signed graph, $G(m,n)$ in the following way. If $n > 0$, $G(m,n)$ is just a rectangular grid graph, i.e., a graph with mn plain vertices arranged into m rows of n vertices each, with plain edges adjoining horizontal and vertical neighbors. Define the conjugate graph of G, dentoted G^*, to be the graph obtained by replacing all of the vertices and vedges of G with their anti-counterparts (assuming that an anti-anti-vertex is a plain vertex, etc.). Under this definition, $(G(m,n))^* = G^*(m,n)$ is a graph with mn anti-vertices arranged into m rows of n vertices with hedges adjoining horizontal neighbors and anti-vedges adjoining vertical neighbors. Now then, the definition of $G(m,n)$ with $n \leq 0$ is just a copy of $G^*(m,n)$ with a column of m empty vertices on the left and m empty vertices on the right. Each empty vertex is connected (with a hedge) only to the anti-vertex horizontal to it (see figure ?).

(TABLE WITH PICTURES OF WEIGHTED MATCHINGS OF $G(2,2)$ AND $G(3,-3)$)

Now for fixed $m > 0$ and any integer n, we define the numbers $M(m,n) = M(G(m,n))$ to be the sum of the weights of the signed matchings of $G(m,n)$. Likewise, $M^*(m,n) = M(G^*(m,n))$ is the sum of the weights of the signed matchings of $G^*(m,n)$.

3 Adjunction

Although it seems there are seperate definitions for $G(m,n)$ when $n > 0$ and when $n \leq 0$, we will show they are actually the same object. In Propp's words, they fit together "seamlessly." We begin by making a natural obseration about the half-infinite sequence $M(G(m,n))$, $n > 0$, and we will prove that it actually holds for any n. In doing so, we will be able to establish that we do indeed

[4] Thinking of monomer-dimer tilings, empty vertices represent places that can only be occupied by a dimer.

have the right definition for $n \leq 0$, and all our desires for the doubly infinite sequence - integrality, uniqueness, reciprocity - will be fulfilled.

(PICTURE OF AN ADJOINED GRAPH OR TWO)

We first notice that for positive n_1, n_2, we can draw the graph $G(m, n_1+n_2)$ by placing $G(m, n_1)$ and $G(m, n_2)$ side by side and using hedges to connect the rightmost m vertices of one to the leftmost m vertices of the other. Call this operation *adjunction*. We write the adjunction of $G(m, n_1)$ and $G(m, n_2)$ as $G(m, n_1)G(m, n_2)$. Since $G(m, n_1)G(m, n_2) = G(m, n_1 + n_2)$, then naturally, $M(G(m, n_1)G(m, n_2)) = M(G(m, n_1 + n_2))$. We'd like to say this is true more generally.

Adjunction:

$$M(G(m, n_1)G(m, n_2) \cdots G(m, n_k))$$
$$= M(G(m, n_1 + n_2 + \cdots + n_k))$$

for all integers $n_1, \ldots, n_k$.

To do so will require two lemmas.

Lemma 1 *The number of matchings of a graph where two vertices are connected by the string: hedge-empty vertex-hedge-empty vertex-hedge, is the same as the number of matchings of the graph where hedge-empty-hedge-empty-hedge is replaced with one hedge.*

Lemma 2 *Let G be a graph where v_1 is a plain vertex with k incident vedges and at most 2 incident hedges, one of which connects v_1 to an empty vertex that is connected (via a hedge) to an anti-vertex of vedge-degree k and at most one other incident hedge (see picture). Let G' be the same as G except that the chain v_1-hedge-empty vertex-hedge-v_2 is replaced by one empty vertex, and further, all the incident vedges of v_1 and v_2 are removed. Then G and G' have the same number of matchings.*

Lemma 1 is an easy bijective proof. Lemma 2 will be proved by the following pictures. The situation described looks something like as follows:

(PICTURES FOR PROVING THE GENERAL BOX PRODUCT)

Notice that most of the matchings of G cancel in pairs. Those that do not immediately cancel are in obvious bijection with the matchings of G'.

Now the adjunction property should be clear. It is easily verified that $M(G(m, 0)G(m, n)) = M(G(m, n))$, so we may assume all the n_i are non-zero. By applying lemma 1 to each place where adjoining takes place, we have that $M(G(m, n_1)G(m, n_2)) = M(G(m, n_1 + n_2)$ whenever n_1 and n_2 are negative (see figure ?). When $n_1 n_2 < 0$, we can apply lemma 2 repeatedly. Upon each application of lemma 2 we change neither the number of matchings nor the difference between the number of vertices and the number of anti-vertices (see figure ?).

(PICTURES SHOWING EACH CASE DESCRIBED ABOVE)

4 Recurrence Relations

As in Propp's paper [1] ...

5 Reciprocity

The reciprocity theorem stated now should be rather anti-climactic. Because of the definition of $G(m,n)$ for $n \leq 0$ and the neediness of empty vertices, we can quickly observe

Reciprocity:

$$M(G(m,-n-2)) = M(G^*(m,n)) = M^*(G(m,n))$$

(see figure ?).

(PICTURE OF RECIPROCITY)

This may seem a bit weak, but it is the natural generalization of the property of dimer coverings observed by Propp [1]. Indeed, if we were to consider the perfect matchings of our graphs, then the statement would be the same:

$$N(G(m,-n-2)) = N(G^*(m,n)) = N^*(G(m,n)).$$

However the sign of every perfect matching of $G^*(m,n)$ is the same, and so $|N(m,n)| = |N^*(m,n)|$. With less-than-perfect matchings, this is clearly not the case.

Perhaps the strength of this result is more obvious when we give all the edges and vertices formal weights. Write the rational generating function

$$R_m(x) = \sum_{n=1}^{\infty} f_n x^n$$

where f_n is a polynomial in the edge and vertex weights of $G(m,n)$. Each term of f_n is defined to be the weight of a matching of $G(m,n)$. (See that setting all the edge and vertex weights equal to 1 gives $f_n = M(m,n)$.) So then considering the reciprocal function(?)

$$\widehat{R}_m(1/x) = f_0 + R_m(1/x) = \sum_{n=0}^{\infty} f_{-n} x^{-n}$$

gives another way to extrapolate to negative values of f_n, and hence of $M(m,n)$. A true "reciprocity" result would be to say $R_m(x) \sim \widehat{R}_m(1/x)$ in some sense.[5] Our result says that although $|M(m,-n-2)| = |M^*(m,n)| \neq |M(m,n)|$, the polynomials f_{-n-2} and f_n do have the same number of terms. (can we say a bit more?)

[5] When f_n encodes perfect matchings, one observes that $Q_m(x) = \pm Q_m(1/x)$ where $R_m = P_m/Q_m$.

6 Box Products

We'd now like to note quickly how the results generalize to $G \boxtimes P_n$. Interpret the box-product graph $H_n = G \boxtimes P_n$ with $n > 0$ as follows: Picture n copies of G lined up side by side. Each vertex in one copy of G is connected with an edge to its image in the copies of G to the left and right of it. Now we'll say that all the edges within G are vedges, and the edges between copies of G are hedges. With this idea, we can define H_{-n} to be H_n^* with m (= the number of vertices in G) empty vertices to the left and to the right of H_n^*. Each empty vertex is connected to exactly one of the vertices of G^* with a hedge.

(PICTURE OF H_n and H_n^*)

Then by way of the lemmas in section 3 (Adjunction), we can show that the adjunction property holds and that there is a reciprocity of the form

$$M(H_{-n-2}) = M(H_n^*) = M^*(H_n).$$

7 More on Linear Recurrences and Reciprocity

Having a reciprocity for box-product graphs is useful when examining matchings on cylinders of fixed circumference and varying height, for example. However some interesting graphs are notably absent from the box-product family: cylinders with fixed height and varying circumference, Möbius strips, tori, and projective planes.

In an unpublished paper [2], Speyer developed a matrix method for encoding perfect matchings of a graph. With this method he was able to state many of the same theorems about recurrences and reciprocity for a broader range of graphs than those we have handled here. In particular, he was able to make a statement about the Möbius strip and projective plane. An adaptation of his method to general matchings seems promising, though experimentation has shown it to be less than straightforward.

As a final thought, we'd like to highlight a key difference between Speyer's method and ours. While we have utilized adjunction as a method for "sticking together" graphs, he employed a different method, called *concatenation*. For simplicity, consider

$$G(m, n_1)G(m, n_2)$$

with $n_1, n_2 > 0$. With adjunction, this means you line up the graph $G(m, n_1)$ alongside $G(m, n_2)$ and connect them with hedges. With concatenation, this means you identify the rightmost vertices of $G(m, n_1)$ with the leftmost vertices of $G(m, n_2)$. So with adjunction $G(m, n_1)G(m, n_2) = G(m, n_1 + n_2)$ whereas $G(m, n_1)G(m, n_2) = G(m, n_1 + n_2 - 1)$ under concatenation. This different approach seems more different when we assign formal weights to edges and vertices. What weights do you give to the overlapping edges? (what to say about this precisely?)

8 Acknowledgments

We'd like to thank...

References

[1] James Propp, *A reciprocity theorem for domino tilings*, The Electronic Journal of Combinatorics **8** (2001) # R18.

[2] David Speyer (?)

[3] Richard Stanley (?)