Games of No Strategy and Low-Grade Combinatorics

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Mathematical Enchantments (mathenchant.org)

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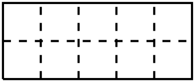
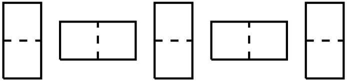
Slides at http://jamespropp.org/moves15.pdf

Interesting math from **boring puzzles**

Boring puzzle: Tile a 2-by-n rectangle (here $n = 5$)



with n dominos.



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Interesting problem: In how many ways can a 2-by-n rectangle be tiled by dominos?

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Answer (old result; whose?): The nth Fibonacci number.

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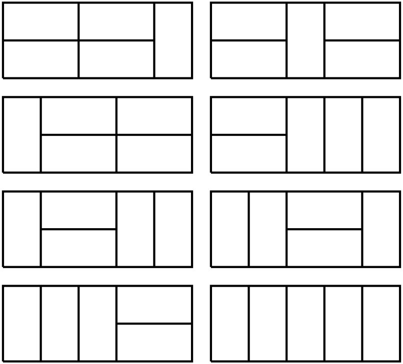
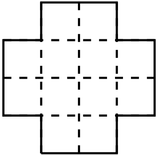
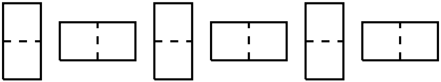


More interesting math from **boring puzzles**

Boring puzzle: Tile an "Aztec diamond of order n" (here $n = 2$)



with $n(n+1)$ dominos.



(Aside: An Aztec diamond of order n consists of rows of length $2, 4, 6, \ldots, 2n-2, 2n, 2n, 2n-2, \ldots, 6, 4, 2$, all centered.)

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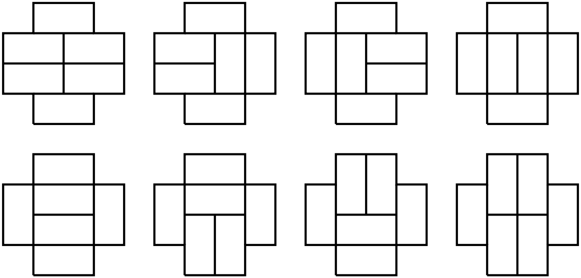

Grades of enumerative combinatorics

These results are part of the branch of mathematics called **enumerative combinatorics**.

I like to split the subject into "grades", according to the growth-rates of the functions involved.

I say $f(n)$ is of **grade** k if $f(n)$ grows like $\exp n^k$ up to smaller correction factors; more precisely, if $(\log\log f(n))/\log n \to k$.

The sequence of Fibonacci numbers is of grade 1; the sequence whose nth term is $2^{n(n+1)/2}$ is of grade 2.

0,1,2,. . . ?

Combinatorics of grade 0 (e.g., figurate numbers) is ancient.

Combinatorics of grade 1 (e.g., Fibonacci numbers, Catalan numbers) goes back several centuries.

Combinatorics of grade 2 is just over a century old.

Aside from a theorem of Linial's on generalized spanning trees in hypergraphs, and a handful of related results, enumerative combinatorics has not progressed beyond grade 2.

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Is there such a thing as enumerative combinatorics of **grade 3**?

If so, how can we "graduate" to it?

Let's return to these questions later.

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Legal move: Join two free ends with a curve that does not cross any previously drawn curve (including the original circle) and then put a short stroke across the curve to create two new free ends, one on either side of the new curve. Delete the old free ends.

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Winner: The last player to make a legal move wins.

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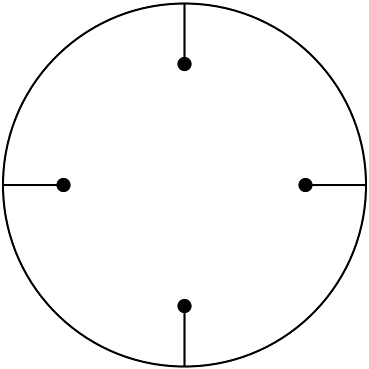

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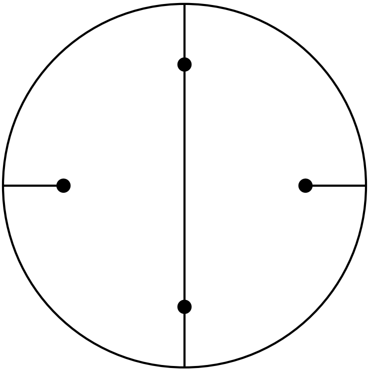

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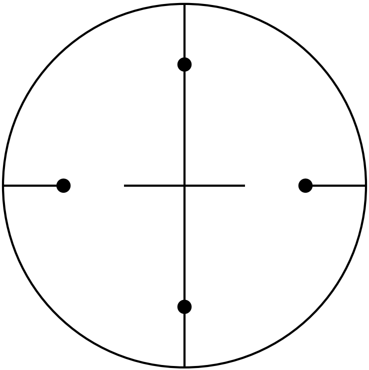

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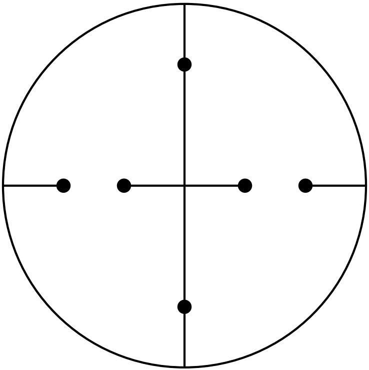

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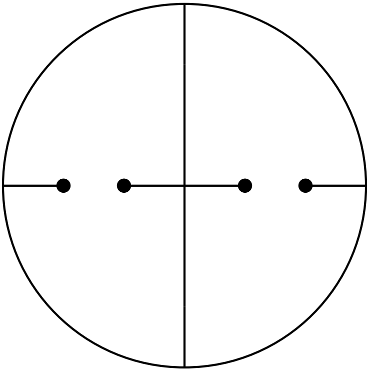

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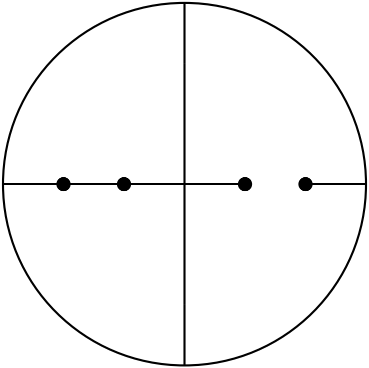

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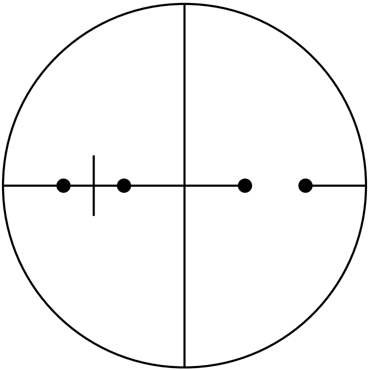

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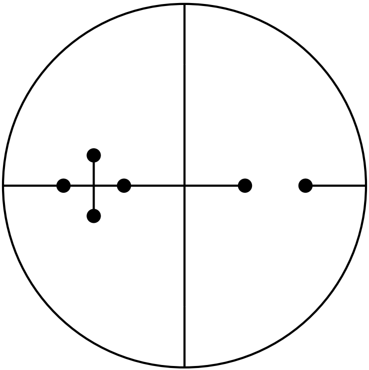

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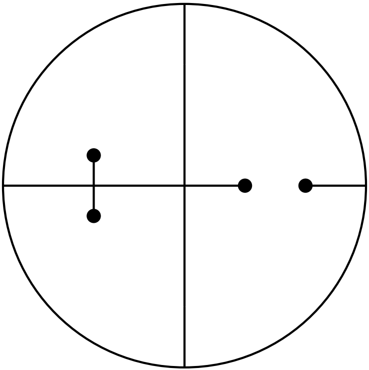

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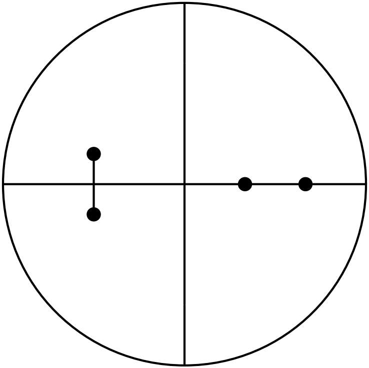

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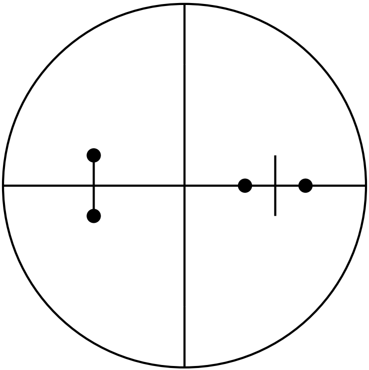

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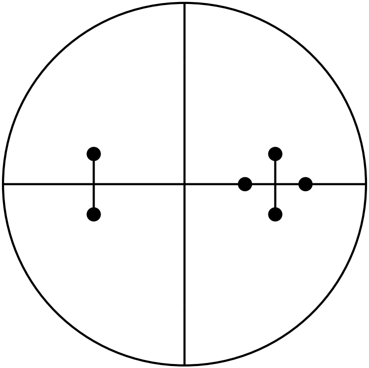

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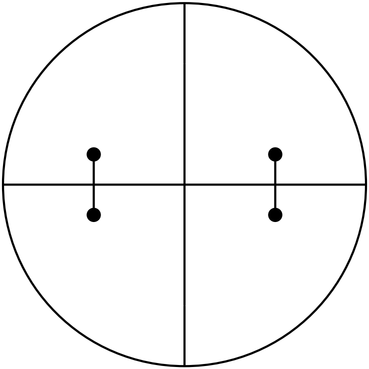

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Question: Is there an exact formula for the number of end-positions of the game?

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Initial state: A row of $2n+1$ bins. The middle bin has 2 chips. Every other bin contains 1 chip, except for the two outermost bins, which are empty.

$$0 \quad 1 \quad 1 \quad \ldots \quad 1 \quad 1 \quad 2 \quad 1 \quad 1 \quad \ldots \quad 1 \quad 1 \quad 0$$

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First player moves:

0	**2**	0	2	0

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$(n^2)!/((1)(2)\cdots(n))((2)(3)\cdots(n+1))\cdots((n)(n+1)\cdots(2n-1))$
(grade 2).

Note that this number is highly composite: it is on the order of $\exp n^2$, but all its prime factors are less than n^2.

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Enumerative combinatorics guild secret: These games are just square standard Young tableaux in disguise.

There's a general theory of chip-firing games, and it predicts that under very general conditions, the number of moves is predestined. But nobody has looked at the number of lines of play.

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The resulting sequence

$$1, 1, 1, 2, 4, 252, 2304, 343712160, 17361257184, \dots$$

just got added to the OEIS a few days ago.

“Welcome to the third grade!” (?)

What makes this sequence so intriguing is a combination of properties:
(1) It's of the THIRD grade.
(2) The terms have lots of small prime factors I can't explain.

$252 = 2^2 \times 3^2 \times 7^1$
$2304 = 2^8 \times 3^2$
$34371260 = 2^5 \times 3^4 \times 5^1 \times 11^1 \times 2411^1$
$17361257184 = 2^5 \times 3^2$ times a big prime
the next term $= 2^7 \times 3^1 \times 11^1 \times 13^1 \times 79^1$ times a big prime
the next term $= 2^4 \times 5^1 \times 17^1 \times 43^1 \times 97^1$ times a product of three big primes

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Statistical physicists have found exact solutions to lots of lattice models (such as tiling models) in 1D and 2D, but not 3D.

Perhaps we could break this dimension barrier if we knew more about grade 3 combinatorics.

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Winner: Last player to make a legal move

That is, the player who uses up all remaining score-lines wins.

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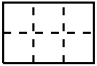
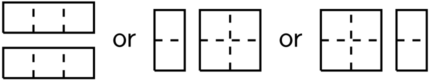


a player can move to



In the first case, we use up 3 lengths of score-line; in the second and third cases, we use up 2 lengths of score-line.

Ignore this slide!

Claim: If a or b is even, a-by-b is a win for the first player.

Proof: Go first, and divide the rectangle into two identical pieces; thereafter, mimic your opponent's move.

You can abuse yourself by working out a winning strategy for the second player when a and b are both odd.

Ignore this slide!

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Proof: Go first, and divide the rectangle into two identical pieces; thereafter, mimic your opponent's move.

You can abuse yourself by working out a winning strategy for the second player when a and b are both odd.

But you shouldn't, because it turns out that EVERY move is a win!

To see why, it's best to ignore everything I've told you in the last two minutes, because it was all designed to distract you from what's really going on.

But don't ignore this one

There's a one sentence-proof that the duration of the game is independent of the moves that are made.

You might want to work on this during the break. If you can't solve the problem, ask a child.

As for the number of lines of play: High school students Caleb Ji, Robin Park, and Angela Song, under the supervision of Tanya Khovanova, and with assistance from Pavel Etingof, have studied the case of a 2-by-n bar.

Letting B_n denote the number of lines of play, they showed that if p is 2, 5, or a prime that is congruent to 1 or 4 mod 5, then

B_n is divisible by p for all sufficiently large n,

whereas if p is ANY other prime, then this is NOT the case.

Further reading

R.J. Anderson, L. Lovász, P.W. Shor, J. Spencer, E. Tardos, and S. Winograd, Disks, balls, and walls: analysis of a combinatorial game, American Mathematical Monthly, Volume 96, Number 6, June-July 1989, pages 481–493.

Grant Cairns and Korrakot Chartarrayawadee, Brussels Sprouts and Cloves, Mathematics Magazine, Volume 80, Number 1, February 2007, pages 46–58(13).

Richard Guy, She Loves Me, She Loves Me Not: Relatives of two games of Lenstra, https://oeis.org/A006016/a006016_1.pdf.

Gil Kalai, Enumeration of Q-acyclic simplicial complexes, Israel J. Math. 45 (1983), no. 4, 337351.

James Propp, Games of no strategy, in preparation.

Slides at http://jamespropp.org/moves15.pdf