18.318: Problem Set #3 (due April 18)

1. Find a q-weighting for perfect matchings of the set $\{1, 2, ..., 2n\}$ such that the sum of the q-weights of the $1 \cdot 3 \cdot 5 \cdots (2n-1)$ perfect matchings is $[1] \cdot [3] \cdot [5] \cdots [2n-1]$.

Solution: Given a (perfect) matching of $\{1, ..., 2n\}$, say that the triple i, j, k $(1 \leq i < j < k \leq 2n)$ is an "inversion" iff i is paired with k and j is paired with something less than k. Every matching of $\{1, ..., 2n\}$ can be converted into a matching of $\{1, ..., 2n-2\}$ by removing the vertex $2n$ and the vertex $2n-1-i$ it is paired with $(0 \leq i \leq 2n-2)$ and re-indexing every vertex j with $j > 2n-1-i$ as vertex $j-1$. Note that in so doing, we eliminate i inversions. Note also that the process is fully reversible, so that a matching of $\{1, ..., 2n-2\}$ becomes $2n-1$ different matchings of $\{1, ..., 2n\}$, having 0, 1, 2, ..., $2n-1$ extra inversions, respectively. Hence the weight enumerator for matchings of $\{1, ..., 2n\}$ equals the weight enumerator for matchings of $\{1, ..., 2n-2\}$ times $1 + q + q^2 + ... + q^{2n-2} = [2n-1]_q$. The claim now follows by induction.

2. Prove that for $m \geq 0$ and $n < 0$,

$$\sum_{i=0}^{n}{}^{L} \begin{bmatrix} i \\ m \end{bmatrix} = q^m \begin{bmatrix} n \\ m+1 \end{bmatrix}.$$

Solution: Recall that

$$\sum_{i=s}^{t}{}^{L} f(i) = -\sum_{i=t}^{s}{}^{L} f(i) = -\sum_{i=t}^{s-1} q^i f(i)$$

for $s > t$, and that $[-k]_q = -[k]_q / q^k$. By use of these facts, one can rewrite the left hand side of the given equation as

$$(-1)^{m+1} \sum_{j=1}^{-n} \begin{bmatrix} j+m-1 \\ m \end{bmatrix}_q / q^{j(m+1)+m(m-1)/2}$$

(using $j = -i$) and the right hand side as

$$(-1)^{m+1} \begin{bmatrix} -n+m \\ m+1 \end{bmatrix}_q / q^{-nm-n+m(m-1)/2} \quad .$$

Equating the two expressions, and replacing $-n$ by N, we get the formula

$$\sum_{j=1}^{N} q^{(N-j)(m+1)} \begin{bmatrix} j+m-1 \\ m \end{bmatrix}_q = \begin{bmatrix} N+m \\ m+1 \end{bmatrix}_q$$

as the identity to be proved. The right hand side q-counts the shortest-length lattice paths from $(0,0)$ to $(m+1, N-1)$, with q-weight equal to the area above and to the left of a path (within the rectangle $[0, m+1] \times [0, N-1]$). Given such a lattice path, choose j as small as possible so that $(m+1, j-1)$ lies on the path. Then there are $N-j$ rows of the rectangle that lie entirely above and to the left of $(m+1, j-1)$, and each such row has length $m+1$. Note also that the predecessor of $(m+1, j-1)$ along the path must be $(m, j-1)$, so that the total q-weight of the path is $q^{(N-j)(m+1)}$ times the q-weight of the partial path from $(0,0)$ to $(m, j-1)$. Summing over all such partial paths, we get the left hand side of the identity to be proved. The desired equality follows.

(Note that there is also a simple algebraic proof: take the original formula for two successive values of n and subtract, obtaining a relation involving three Gaussian polynomials with negative upper index. Clearing out all common factors, one is left with a simple q-polynomial identity that is easily verified.)

3. Let $f(q)$ be the weight-enumerator of partitions in which all parts are distinct and differ by at least 3, and let $g(q)$ be the weight-enumerator of partitions that satisfy the same constraint, with the additional condition that parts that are multiples of 3 must differ by at least 6. Calculate $f(q)$ and $g(q)$ up to degree 18. Does $f(q)$ appear to be given by a nice product formula? What about $g(q)$?

Solution: The coefficients of $f(q)$ up to degree 18 are

$$1, 1, 1, 1, 1, 2, 2, 3, 3, 4, 4, 5, 6, 7, 8, 10, 11, 13, 15$$

while those of $g(q)$ are

$$1, 1, 1, 1, 1, 2, 2, 3, 3, 3, 4, 5, 6, 7, 8, 9, 10, 12, 14.$$

The former q-series is equal to

$$(1-q^1)^{-1}(1-q^5)^{-1}(1-q^7)^{-1}(1-q^9)^{-1}(1-q^{10})^{+1}(1-q^{11})^{-1}\times$$
$$(1-q^{13})^{-1}(1-q^{14})^{+1}(1-q^{15})^{-2}(1-q^{16})^{+1}(1-q^{17})^{-2}(1-q^{18})^{+1},$$

which shows no strong signs of pattern, while the latter q-series is equal to

$$(1-q^1)^{-1}(1-q^5)^{-1}(1-q^7)^{-1}(1-q^{11})^{-1}(1-q^{13})^{-1}(1-q^{17})^{-1}$$

which suggests the conjecture

$$g(q) = \prod_{n=1}^{\infty}(1-q^{6n-5})^{-1}(1-q^{6n-1})^{-1}.$$

This is indeed true, and is called Schur's identity.

4. Call an inverted triangular array of integers and half-integers *permissible* if the entries in the odd-length rows are integers and the entries in the even-length rows are half-integers (i.e., halves of odd integers), and if the local configuration

$$\begin{array}{ccc} a & & a' \\ & b & \end{array}$$

can only occur if $a+\frac{1}{2} \le b \le a'-\frac{1}{2}$. Call such an array *symmetric* if each of its rows $(a_1, a_2, ..., a_k)$ has the property that $a_i + a_{k+1-i} = 0$ for all $1 \le i \le k$; e.g., the array

$$\begin{array}{ccccccccc} -4 & & -2 & & 0 & & +2 & & +4 \\ & -\frac{7}{2} & & -\frac{3}{2} & & +\frac{3}{2} & & +\frac{7}{2} & \\ & & -3 & & 0 & & +3 & & \\ & & & -\frac{1}{2} & & +\frac{1}{2} & & & \\ & & & & 0 & & & & \end{array}$$

is a symmetric permissible array. Find general formulas for $O(a_1, a_2, ..., a_n)$, the number of symmetric permissible arrays with top row $(-a_n, ..., -a_2, -a_1, 0, a_1, a_2, ..., a_n)$, and for $E(a_1, a_2, ..., a_n)$, the number of symmetric permissible arrays with top row $(-a_n, ..., -a_2, -a_1, a_1, a_2, ..., a_n)$.

Solution: Let $\sum_{i=s}^{\prime\, t} f(i) = f(s+\frac{1}{2}) + f(s+\frac{3}{2}) + \ldots + f(t-\frac{1}{2})$, and let $\langle s_1, t_1; s_2, t_2; \ldots; s_n, t_n | f(b_1, b_2, \ldots, b_n)\rangle$ denote

$$\sum_{b_1=s_1}^{t_1}{}' \sum_{b_2=s_2}^{t_2}{}' \cdots \sum_{b_n=s_n}^{t_n}{}' f(b_1, b_2, \ldots, b_n).$$

The operator $\sum'$, suitably extended to allow $s \geq t$, has all the nice properties that $\sum^L$ has, as studied in class, e.g.,

$$\sum_{i=s}^{t}{}' + \sum_{i=t}^{u}{}' = \sum_{i=s}^{u}{}'.$$

Using this operator, we can extend the definitions of $E(\cdot)$ and $O(\cdot)$ by putting $E(a_1) = O(a_1) = 1$ and recursively defining

$$O(a_1, a_2, ..., a_n) = \langle 0, a_1; a_1, a_2; \ldots; a_{n-1}, a_n | E(b_1, b_2, \ldots, b_n) \rangle$$

and

$$E(a_1, a_2, ..., a_n) = \langle a_1, a_2; a_2, a_3; \ldots; a_{n-1}, a_n | E(b_1, b_2, \ldots, b_{n-1}) \rangle.$$

Using these extended definitions, we will prove by induction that the functions $E(a_1), O(a_1), E(a_1, a_2), O(a_1, a_2), \ldots$ are alternating functions of their variables, and that the E's are even functions while the O's are odd functions. As a base case, we have $E(a_1) = 1$. Now suppose $E(b_1, b_2, \ldots, b_n)$ is alternating and even. Then

$$\begin{aligned} O(-a_1, a_2, ..., a_n) &= \langle 0, -a_1; -a_1, a_2; \ldots; a_{n-1}, a_n | E(b_1, b_2, \ldots, b_n) \rangle \\ &= \quad \langle 0, -a_1; -a_1, 0; \ldots; a_{n-1}, a_n | E(b_1, b_2, \ldots, b_n) \rangle \\ &\quad + \langle 0, -a_1; 0, a_1; \ldots; a_{n-1}, a_n | E(b_1, b_2, \ldots, b_n) \rangle \\ &\quad + \langle 0, -a_1; a_1, a_2; \ldots; a_{n-1}, a_n | E(b_1, b_2, \ldots, b_n) \rangle. \end{aligned}$$

The first term on the right can be written as

$$-\langle 0, -a_1; 0, -a_1; \ldots | E(b_1, b_2, \ldots) \rangle,$$

which vanishes because of the skew-symmetry of $E(b_1, b_2, \ldots)$, while the second term (afer the change in variables $b_1 \to -b_1$) becomes

$$\langle a_1, 0; 0, a_1; \ldots | E(b_1, b_2, \ldots) \rangle,$$

which vanishes just as the first term did. Hence we are left with

$$\begin{aligned} & \langle 0, -a_1; a_1, a_2; \ldots | E(b_1, b_2, \ldots) \rangle \\ = & \langle a_1, 0; a_1, a_2; \ldots | E(-b_1, b_2, \ldots) \rangle \\ = & \langle a_1, 0; a_1, a_2; \ldots | E(b_1, b_2, \ldots) \rangle \quad E(\cdot) \text{ is even)} \\ = & -\langle 0, a_1; a_1, a_2; \ldots | E(b_1, b_2, \ldots) \rangle \\ = & -O(a_1, a_1, \ldots, a_n). \end{aligned}$$

Thus $O(a_1, \ldots, a_n)$ is odd in a_1. The proof that $O(a_1, \ldots, a_n)$ is alternating is the same as seen in class, so we omit it. Oddness in a_1 plus skew-symmetry in all the variables implies oddness in all the variables. Similarly, assuming that $O(\cdot)$ is odd and alternating, we can use

$$E(a_1, a_2, \ldots, a_n) = \langle a_1, a_2; a_2, a_3; \ldots; a_{n-1}, a_n | O(b_1, b_2, \ldots, b_{n-1}) \rangle$$

to show that $E(\cdot)$ is even and alternating. Thus $E(a_1, \ldots, a_n)$ and $O(a_1, \ldots, a_n)$, viewed as polynomials in $a_1, ..., a_n$, must be multiples of $\prod_{i<j}(a_j^2 - a_i^2)$ and $a_1 a_2 \cdots a_n \prod_{i<j}(a_j^2 - a_i^2)$, respectively.

On the other hand, the recurrence relation gives us leading terms $1, a_1, \frac{1}{2}a_2^2, \frac{1}{6}a_1 a_2^3, \frac{1}{48}a_2^2 a_3^4, \frac{1}{600}a_1 a_2^3 a_3^5, \ldots$. Hence

$$E(a_1, \ldots, a_n) = \frac{1}{2!4! \cdots (2n-2)!} \prod_{1 \leq i < j \leq n} (a_j^2 - a_i^2)$$

and

$$O(a_1, \ldots, a_n) = \frac{1}{1!3! \cdots (2n-1)!} \prod_{1 \leq i \leq n} a_i \prod_{1 \leq i < j \leq n} (a_j^2 - a_i^2).$$

5. Show that the number of rhombus tilings of an $a, 2b, a, a, 2b, a$ hexagon that are bilaterally symmetric about the axis joining the midpoints of the two sides of length $2b$ is given by the expression

$$\prod_{i=1}^{a} \prod_{j=1}^{i-1} \prod_{k=1}^{b} \frac{i+j+2k-1}{i+j+2k-3}.$$

(What about an $a, 2b+1, a, a, 2b+1, a$, hexagon?)

Solution: Consider a hexagon with a symmetric tiling. Since there is a single rhombus that lies in the top two rows of the hexagon, that rhombus must be centered, implying that the length of the top edge of the hexagon must be even.

A slight variation of the bijection used in class shows that the symmetric tilings are in one-to-one correspondence with the symmetric arrays with top row $(-a-b+\frac{1}{2}, -a-b+\frac{3}{2}, \ldots, -b-\frac{1}{2}, b+\frac{1}{2}, \ldots, a+b-\frac{3}{2}, a+b-\frac{1}{2})$, so that the number of symmetric tilings is

$$E(b+\frac{1}{2}, b+\frac{3}{2}, \ldots, a+b-\frac{1}{2})$$

$$\begin{aligned}
&= \frac{1}{2!4!\cdots(2n-2)!}\left(\prod_{1\le i<j\le a}(b-\frac{1}{2}+j)-(b-\frac{1}{2}+i)\right)\times \\
&\quad \left(\prod_{1\le i<j\le a}(b-\frac{1}{2}+j)+(b-\frac{1}{2}+i)\right) \\
&= \frac{1}{2!4!\cdots(2n-2)!}\left(\prod_{1\le i<j\le a} j-i\right)\left(\prod_{1\le i<j\le a} 2b-1+i+j\right) \\
&= \frac{1}{2!4!\cdots(2n-2)!}(1!2!\cdots(a-1)!)\prod_{2\le j\le a}\frac{(2b+2j-2)!}{(2b+j-1)!} \\
&= \frac{1}{2!4!\cdots(2n-2)!}(1!2!\cdots(a-1)!)\frac{(2b+2)!(2b+4)!\cdots(2b+2a-2)!}{(2b+1)!(2b+2)!\cdots(2b+a-1)!}.
\end{aligned}$$

When you divide this by $E(b+\frac{1}{2}, b+\frac{3}{2}, \ldots, (a-1)+b-\frac{1}{2})$, you get

$$\begin{aligned}
&\frac{1}{(2a-2)!}(a-1)!\frac{(2b+2a-2)!}{(2b+a-1)!} \\
&= \frac{(2b+a)!(2b+a+1)!\cdots(2b+2a-2)!}{(a)!(a+1)!\cdots(2a-2)!} \\
&= \prod_{j=1}^{a-1}\frac{(a+2b-1+j)!}{(a-1+j)!}.
\end{aligned}$$

So, multiplying all these factors together, we get

$$\prod_{i=1}^{a}\prod_{j=1}^{i-1}\frac{(i+2b-1+j)!}{(i-1+j)!},$$

which expands to

$$\prod_{i=1}^{a}\prod_{j=1}^{i-1}\prod_{k=1}^{b}\frac{i+j+2k-1}{i+j+2k-3},$$

as desired.