18.318 Topics in Combinatorics Spring 1995

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Tu-Th 9:30-11:00, room 2-151

Course description

The course as a whole will be an introduction to advanced combinatorics, with an emphasis on bijective, algebraic, and linear-algebraic techniques. To some extent it will be an assortment of various topics that I regard as important and/or beautiful. But there'll also be an over-arching goal throughout the term: to accumulate a sufficient amount of background to enable us to end the course by reading Greg Kuperberg's recent article describing his enumeration of several symmetry-classes of plane partitions (including one hitherto-unsolved case). Since this research (like the even more difficult research of Doron Zeilberger on alternating-sign matrices) was conducted with extensive help from computer algebra systems, my lectures will also include an introduction to techniques for the automated proof of combinatorial identities.

I'll begin the course with a couple of gems: Rota's article on Bell numbers and Lovász's determination of the Shannon capacity of the pentagon. Both use linear algebra in imaginative ways to solve combinatorial problems. Then we'll do a little bit of Pólya theory, and introduce the Möbius function to help us enumerate colored necklaces. We'll study Möbius inversion in more general posets and explore another pretty result of Lovász, in which Möbius inversion is used to prove a result about sorting. We'll also look at distributive lattices and order-ideals of posets.

The study of order-ideals will lead us into the theory of partitions of numbers (in the sense of Euler), which will in turn lead us to the idea of q-analogues, and in particular, the q-binomial coefficients. We'll study various partition identities, such as the Jacobi triple product identity and the Rogers-Ramanujan identities, seeing how both algebraic and bijective proofs have roles to play. One of our approaches to the Rogers-Ramanujan identities will reduce the proof of these identities to certain identities of a more tractable kind. These latter identities are now susceptible to a routinized style of proof, so this will serve as a transition to a brief study of the Wilf-Zeilberger method of certifying identities.

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Then we'll switch gears and study a different twist on binomial coefficients, in which the number on top can be negative. We'll see how these numbers are governed by a conceptually satisfying reciprocity law relating them to the normal binomial coefficients. We'll also see how this lets us attach a meaning to the value of the chromatic polynomial $\chi(n)$ when n is a negative integer.

Then we'll switch gears again and start to think about matrices from various combinatorial points of view. First, we'll see how entries in powers of non-negative integer matrices enumerate walks in directed graphs; by incorporating indeterminates into the matrices, one can get very slick proofs of otherwise hard-to-prove enumerative facts. A related technique is the transfer matrix method, of which we'll study one or two applications. Then we'll turn the tables and try to see what insights combinatorics can bring to the study of algebra; in particular, we'll look at combinatorial proofs of the Vandermonde determinant formula (in which determinants count tournaments) and the Cayley-Hamilton Theorem. After a day or two of looking at the matrix tree theorem and related results (in which determinants count spanning arborescences), we'll study commutation monoids and heaps, if time permits.

We will then study the work of Gessel and Viennot, in which determinants are interpreted as counting non-intersecting families of lattice-paths, and the work of Kasteleyn, in which determinants and Pfaffians are interpreted as counting perfect matchings. It has recently emerged that in a deep sense these two techniques are the same! Once we've come to appreciate this, we'll be in a position to study ≥ 3 different proofs of MacMahon's formula for plane partitions: the Bender-Knuth bijective proof (based on the Schensted correspondence), an inductive method based on a trick I learned from Noam Elkies, and the matchings-proof put forward by Greg Kuperberg. From this last proof, we'll proceed to a study of various symmetry-classes of plane partitions, and (if we're lucky) make it most of the way through Greg's article.

Since is the first time I have taught such a course, I may not end up covering all of the preceding material, but I intend to try.

There will be a homework assignment roughly every three weeks. Students taking the course for credit will be expected to attend regularly and to hand in the homework.