18.177: Problem Set #2 (solutions)

1. (a) Given $B_1, B_2 \subseteq A$ with $\mu(B_1)$, $\mu(B_2) > 0$, ergodicity of T implies the existence of n such that $\mu(T^n B_1 \cap B_2) > 0$. Every point in $T^n B_1 \cap B_2$ is in $T^n B_1 \cap A$ and hence is in one of the sets $T_A^m B_1$ ($m = 1, 2, ...$). Since there are only countably many m's, one of these sets $T_A^m B_1$ must intersect $T^n B_1 \cap B_2$ in a set of positive measure. This yields m such that $T_A^m B_1 \cap B_2$ has positive measure. Since B_1, B_2 were arbitrary, T_A is ergodic. (b) View the extended space as consisting of finite fibers (of varous sizes) over the base-space X; the map sends the bottom point in each fiber to the one above it, and so on. Any T_f-invariant subset A of the extended space projects to an invariant subset B of X. Ergodicity of T implies that B has either measure 0 or measure 1, and in either case we easily conclude that A has the same property.

2. First we show that for any set A of positive measure, any positive integer m, and any $0 < \epsilon \leq 1$, there exists a set A' with $\mu(A \triangle A') \leq \epsilon$ containing a positive-measure set of points x with $n_{A'}(x) \geq m$. For, given A, m, and ϵ, we may take an arbitrary set B of measure $\delta = \epsilon/m$. Since T is ergodic, the expected return-time from B to itself is $m/\epsilon \geq m$, so that in particular there must be a positive-measure subset $B' \subseteq B$ on which the return-time to B is $\geq m$. Let $A' = A \cup B' \setminus (TB' \cup T^2 B' \cup \ldots T^{m-1} B')$. Since $\mu(B') \leq \mu(B) = \epsilon/m$, $\mu(A' \setminus A) \leq \epsilon$, and it is easily seen that for $x \in B'$, $n_{A'}(x) \geq m$.

Now we use this fact to solve the problem. Given an initial set A_0, we construct sets A_1, A_2, etc., so that for each k, $\{x : n_{A_k} > k\}$ has positive measure c_k, and such that $\{x : n_{A_l} > k\}$ has measure at least $\frac{1}{2} c_k$ for all $l \geq k$ (for each l, this just means choosing the perturbation ϵ_l subject to a finite number of limitations on its size). If we further arrange so that ϵ_k goes to zero exponentially (and hence summably), we obtain in the limit at set A^* such that $n_{A^*}(x)$ takes on infinitely many values.

(One can also give solutions in which one starts with a Rokhlin base and whittles it down, rather than building it up. One must take care that the Rokhlin base itself, along with certain other sets, do not tend to zero in measure as the iterative construction proceeds.)

3. Given a positive integer k, there exists a positive integer N such that every integer $\geq N$ can be written as a sum of k's and $k+1$'s (in fact, $N = (a-1)(b-1)$ will do and is best possible). Taking a set B of measure

$< 1/N$ as an initial attempt at a Kakutani base, we find that there must be a positive-measure subset B' of B on which $n_B(x) > N$. Taking B' as our new base, we have $n_{B'}(x) > N$ for almost every $x \in B'$. Hence in particular the sets $T^i B'$ $(0 \leq i \leq N-1)$ are disjoint. Divide the Kakutani tower with base B' into "columns", where x, x' belong to the same column iff they lie above points in B' with the same value of $n_{B'}$. Within each column we may select a subset of the levels (starting with the bottom level) so that all gaps between nearest selected levels, including the gap between the last selected level and the return to B', is of length k or $k+1$.

4. Fix $\lambda \in \mathbf{C}$, $|\lambda| = 1$. Given a Rokhlin tower with n levels that covers all but $1/n$ of the measure of the space, we can define f_n to be λ^k on the kth level of the tower $(0 \leq k \leq n-1)$ and 1 on the left-over set, so that $||f_n||_2 = 1$. Then $f_n(Tx) = \lambda f_n(x)$ for all x outside a set of measure $2/n$ (namely the top level of the tower and the left-over set), and on this small set both side of the equation are bounded in absolute value by 1. Hence $(||Uf_n - \lambda f_n||_2)^2 = \int_X |f_n(Tx) - \lambda f_n(x)|^2 \, d\mu \leq 2/n \to 0$.

5. Fix $n \geq 2/\epsilon$. In each space, we can build a Rokhlin tower with n levels $(n \geq 2/\epsilon)$ that covers exactly $1-\epsilon/2$ of the measure of the space. Choose some measure-preserving bijection between the bases of the two towers, and for $1 \leq i \leq n-1$ correspond the ith levels of the two towers so that $T'(\Phi(x)) = \Phi(T(x))$ for all x in the $i-1$st level. Also choose some measure-preserving bijection between the left-over sets in the two spaces. Then $T'(\Phi(x)) = \Phi(T(x))$ for all x outside of the top level of the tower and the left-over set, which jointly have measure $< \epsilon$.

6. Since T sends a cylinder set specified at n consecutive positions to another such cylinder set, and since each such cylinder set has μ-measure 2^{-n}, we see that T preserves the measure of every set in the cylinder algebra; since these sets generate the full Borel σ-algebra, T is measure-preserving. If there were a (non-trivial) T-invariant set, there would have to be a to-within-ϵ T-invariant cylinder set; however, the orbit of every cylinder set of positive measure is the entire space. It follows that T is ergodic.

For all $n \geq 0$ we can decompose the space into 2^n levels according to the n least significant bits; then the map T carries the kth level into the $k+1$st $(0 \leq k < 2^n)$ and carries the $2^n - 1$st level back into the 0th. If we take λ to be a 2^nth root of unity, we can put $f(x) = \lambda^k$ for all x in the kth level,

obtaining an eigenfunction with eigenvalue λ. Note that these eigenfunctions form a complete orthonormal basis for the space (to see this, just think of the strings of bits as binary expansions of numbers in $[0,1]$, always keeping in mind that sets of measure 0 don't matter). Hence the system has discrete spectrum, and its spectrum is the subgroup of K consisting of the 1st, 2nd, 4th, 8th, ... roots of unity.

7. Lemma: The spectrum of irrational rotation by θ $(0 < \theta < 2\pi)$ consists of all the complex numbers of the form $e^{in\theta}$. Proof: The map $z \mapsto z^n$ on the unit circle K in $\mathbf{C}$ gives an L^2-function whose eigenvalue is $e^{in\theta}$. These maps (as always) form a basis for L^2 of the circle, so these eigenvalues constitute the entire spectrum.

Suppose now that α and β are distinct irrational numbers between 0 and 1 that are irrationally independent over $\mathbf{Q}$ in the sense that $m\alpha + n\beta$ can only be an integer when $m = n = 0$. Then the values $e^{(2\pi i\alpha)m+(2\pi i\beta)n}$ are all distinct from one another and from 1 (except for $m = n = 0$), so that the dynamical system whose space is the 2-dimensional torus $K \times K$, whose measure is the product of circle-measure with itself, and whose dynamics are given by the map $T : (x,y) \mapsto (x + \frac{1}{\alpha}, y + \frac{1}{\beta})$, is ergodic. This tells us that for almost all x, y, the fraction of the time that $T^n(x,y)$ spends in $(1 - \frac{1}{\alpha}, 1) \times (1 - \frac{1}{\beta}, 1)$ is $1/\alpha\beta$. Unfortunately, this is not quite enough for our purposes: We need to show that this claim holds for the *specific* point $(x,y) = (0,0)$, which has measure 0.

To handle this problem, we employ the same Fourier argument that we used in class to show that orbits under irrational rotation of the circle are uniformly distributed. One shows that for every initial point (x,y) and for every complex exponential function f on the torus, the average $(\frac{1}{n}(f + f \circ T + ... + f \circ T^{n-1}))(x,y)$ converges to 0 (unless f is the constant 1). Then, using the fact that continuous functions can be approximated arbitrarily well by sums of complex exponentials, and the fact that indicator functions of intervals can be approximated arbitrarily well by continuous functions, we are able to conclude that for every initial point (x,y), including $(0,0)$, the number of n for which $T^n(x,y)$ is in $(1 - \frac{1}{\alpha}, 1) \times (1 - \frac{1}{\beta}, 1)$, divided by n, converges to the measure of this product of two intervals, which is just $1/\alpha\beta$.

(Note: George Homsy's solution dealt with the almost-everywhere versus at-one-point issue by putting a small neighborhood around the initial point, and making use of the rigidity of the rotation map. This works, too.)

8. If T were not weakly mixing, it would have a circle-rotation as a factor. Since factors can be construed as sub-algebras, it would follow that the rotation would also have to satisfy the stated property involving the constant c. However, if we take $A{=}B$ to be an interval of some length $\ell < 1/2c$ in the circle, then we can find infinitely many values of n for which $\mu(T^n A \cap B) > \ell/2 > c\mu(A)\mu(B)$: a contradiction.

Now let m be a cluster point of the measure ν_n. For ease of notation, let us write $\nu_n \to m$, ignoring the fact that we might need to pass to a subsequence. We have $m(A \times B) = \lim m(T^n A \cap B)$ and $m(TA \times TB) = \lim m(T^n TA \cap TB) = \lim m(T(T^n A \cap B))$, so $m(A \times B) = m(TA \times TB)$. It follows that m is invariant under $\overline{T}$, where $\overline{T}$ is defined via $\overline{T}(x, y) = (Tx, Ty)$. (Recall that the measurable rectangles generate the full σ-algebra in the product, so what's true for rectangles is true for all measurable sets.) Since T is weakly mixing, $\overline{T}$ is ergodic.

m has the property that a set with $\mu \times \mu$-measure ϵ has m-measure at most $c\epsilon$. It follows that the measure m is absolutely continuous with respect to the product measure (i.e., any set that gets measure 0 under the product measure also gets measure 0 under m). By our results on ergodic decomposition, m must actually be the product measure itself. Hence the desired convergence occurs.