18.177: Problem Set #1 (solutions)

1. (a) Consider the map Φ that sends $(x_0, x_1, x_2, \ldots)$ to $\sum_{i=0}^{\infty} x_i/2^{i+1}$. Φ is surjective, and it is one-to-one almost everywhere (namely, off the set of dyadic rationals). Since the pre-image of every interval $[j/2^{i+1}, (j+1)/2^{i+1})$ (with $0 \leq j < 2^{i+1}$) is a cylinder set of the same measure as the interval (namely $2^{-(i+1)}$), and since the dyadic intervals (on the one hand) generate the Borel sets in $[0,1)$ while the cylinder sets (on the other hand) generate the Borel sets in $\{0,1\}^{\mathbf{N}}$, we see that Φ preserves measure. Lastly, it is simple to check that $\sum_{i=0}^{\infty} x_{i+1}/2^{i+1}$ is congruent to twice $\sum_{i=0}^{\infty} x_i/2^{i+1}$ modulo 1 (the difference is just x_0).
(b) Consider the map Φ that sends $(\ldots, x_{-1}, x_0, x_1, \ldots)$ to (x, y) where $x = x_0/2 + x_1/4 + x_2/8 + \ldots$ and $y = x_{-1}/2 + x_{-2}/4 + x_{-3}/8 + \ldots$. The (non-dynamical) measure-theoretical equivalence works out just as in part (a). If we apply Φ to the shift of $(\ldots, x_{-1}, x_0, x_1, \ldots)$, we get (x', y') where $x' = x_1/2 + x_2/4 + x_3/8 + \ldots$ and $y' = x_0/2 + x_{-1}/4 + x_{-2}/8 + \ldots$; the former is $2x$ mod 1, while the latter is either $y/2$ or $(y+1)/2$, according to whether x_0 is 0 or 1.

2. Let $A_\epsilon = \{x : f(x) > \epsilon\}$. For any $\epsilon > 0$, the strong form of the Poincaré Recurrence Theorem tells us that almost every point in A_ϵ re-visits A_ϵ infinitely often, so that $\sum_{n=1}^{\infty} f(T^n x)$ diverges for almost every x in A_ϵ. Sending ϵ to zero, we conclude that this holds for almost all x in X. That is, the set of x for which convergence happens has measure 0.

3. Let $N = \{x : f(x) \neq f(Tx)\}$, and let $N' = \{x : T^n x \in N \text{ for some } n \geq 0\}$. N is a null set by hypothesis, and $N' = N \cup T^{-1}N \cup T^{-2}N \cup \ldots$, being a countable union of null sets, is also null. Note that N' is also invariant (see problem 4). Define f' so that $f'(x)$ is 0 for $x \in N'$ and is $f(x)$ otherwise. Then f' agrees with f off of N and is invariant.

4. Let $A' = \cup_{n \geq 0} T^{-n} A$. We have $T^{-1}A' = \cup_{n \geq 1} T^{-n} A \subseteq A'$. The set $A' \setminus T^{-1}A'$ consists of points in A that never return to A, which (by Poincaré recurrence) constitute a null-set. Hence A' is actually invariant modulo null sets. (Alternative argument: Since $T^{-1}A' \subseteq A'$ and $\mu(T^{-1}A') = \mu(A')$, $\mu(T^{-1}A' \triangle A')$ must vanish.) $\mu(A') \geq \mu(A) > 0$, so by ergodicity, $\mu(A')$ must be 1. The second part of the problem is actually subtly wrong as stated, since we don't have hypotheses that guarantee that the sets $T^n A$ are even

measurable. However, *if* they are measurable, the set $A'' = \cup_{n \geq 0} T^n A$ must have measure 1. To see this, note that $T^{-1}A''$ contains A'', so (using the same reasoning as in the alternative solution to the first part) A'' must be invariant modulo a null set, and as before we can conclude that A'' must have measure 1.

5.

$$\begin{aligned}
(Uf, g) &= \int_X (Uf)(\overline{g})\, d\mu \\
&= \int f(Tx)\overline{g(x)}\, d\mu(x) \\
&= \int f(y)\overline{g(T^{-1}y)}\, d\mu(y) \\
&\quad \text{(change of variable and } T\text{-invariance of } \mu) \\
&= \int_X (f)(U^{-1}g)\, d\mu \\
&= (f, U^{-1}g).
\end{aligned}$$

6. (a) It suffices to prove the claim with f equal to the indicator function of an arbitrary interval. That is, for all $a \leq b$ we must show that

$$\int_{a_1}^{b_1} \frac{dx}{1+x^2} + \int_{a_2}^{b_2} \frac{dx}{1+x^2} = \int_a^b \frac{dx}{1+x^2}$$

where $a_1 < a_2$ are the pre-images of a (under the map $x \mapsto \frac{1}{2}(x - \frac{1}{x})$). and $b_1 < b_2$ are the pre-images of b. The claim is clearly true when $b = a$, so it in turn suffices to check that the left and right hand sides have the same derivative as a function of b, with a fixed. The rate of change of the first integral is $\frac{db_1}{db}\frac{1}{1+b_1^2}$, which (via the formula $b_1 = b - \sqrt{b^2+1}$) can be shown to equal $\frac{1}{2}\frac{1}{b^2+1}$. The second integral has the same rate of change, yielding $\frac{1}{b^2+1}$ as the rate of change of the left side. On the other hand, the rate of change of the right side is clearly $\frac{1}{b^2+1}$; hence the claim holds.
(b) The change of variables $x = \tan t$ conjugates the mapping $x \mapsto \frac{1}{2}(x - \frac{1}{x})$ to the mapping $t \mapsto 2t - \frac{\pi}{2}$ (modulo π) and the measure $\frac{dx}{1+x^2}$ to Lebesgue measure dt on $[-\frac{\pi}{2}, \frac{\pi}{2})$.
(c) The change of variables $t = \pi(s + \frac{1}{2})$ preserves Lebesgue measure (up to scaling) and conjugates the mapping $t \mapsto 2t - \frac{\pi}{2}$ (modulo π) to the mapping

$s \mapsto 2s$ (modulo 1), which (as we saw in problem 1) is isomorphic to a one-sided Bernoulli shift.

(d) Put $t = \frac{1}{1+x^2}$. This carries the mapping $x \mapsto \frac{1}{2}(x - \frac{1}{x})$ to the mapping $t \mapsto 4t(1-t)$ and the measure $\frac{dx}{1+x^2}$ to the measure $\frac{dt}{2\sqrt{t(1-t)}}$.

(e) (SKETCH) When $a < 0$, then the upper half plane gets attracted to $i\sqrt{-a}$ under iteration of T_a while the lower half plane gets attracted to $-i\sqrt{-a}$; points on the real line get mapped around chaotically. When $a > 0$, the geometry is no longer as described in the problem; rather, the *imaginary* axis becomes the "equator" on which chaos occurs, torn between the two poles $\sqrt{a}$ and $-\sqrt{a}$. In the degenerate case $a = 0$, all points in $\mathbf{C}$ get sucked into 0.