18.177: Problem Set #4 (due Thursday, May 16)

1. In my discussion of a particular self-coupling of an aperiodic irreducible finite-state Markov chain with itself, I made use of the assertion that $p^{(n)}(u,y) = \text{Prob}[\, X_n = y \,|\, X_0 = u,\ Y_0 = v\,]$ for all u, v, y. Does this equality follow from the fact that the $(X_k, Y_k)_{k=0}^{\infty}$-process is a coupling of the Markov chain with itself (i.e., from the fact that both the $(X_k)_{k=0}^{\infty}$ and $(Y_k)_{k=0}^{\infty}$ marginals follow the probability law of the original Markov chain), or does it use some special property of the particular coupling? Clarify your analysis with proofs/counterexamples as appropriate. (If your notes and the on-line notes for the April 18 lecture are too sketchy, you can look at Theorem 1.2 in Chapter II of Liggett, and the first equality that appears on page 67 in particular; Liggett indicates time-zero conditioning via superscripts.)

2. Show that an i.i.d. measure (taking its values in $\{0,1\}$) is indecomposable in the class of exchangeable measures.

3. Prove that the decomposition of exchangeable 0,1–valued measures into i.i.d. measures is unique.

4. Prove DeFinetti's Theorem for exchangeable measures on doubly-infinite sequences that take their values in any finite set (not just $\{0,1\}$).

5. Suppose we are attempting to define a continuous-time exchange process in d-dimensional space with states in $\{0,1\}^{\mathbf{Z}^d}$ such that updates are exchanges affecting two adjacent sites, and that for this purpose we are using exponential timers of rate 1 associated with all such pairs of adjacent sites, so that when a timer goes off, the two sites swap bits. Prove that the evolution of this system is well-defined, in the sense that with probability 1, one can determine the bit at every site at time t for every $t > 0$.

6. Recall the discrete-time one-dimensional totally asymmetric exclusion process discused in class, where at each time step, a particle with a particle to its right stays put with probability 1, while a particle with a vacancy to its right stays put with probability $\frac{1}{2}$ and moves to the right with probability $\frac{1}{2}$. Conjecture a form for a translation-invariant probability law μ on $X =$

$\{0,1\}^{\mathbf{Z}}$ with $\mu(\{\eta \in X : \eta(0) = 1\}) = \frac{1}{2}$ that is stationary under this evolution, *other* than the two-point measure that assigns probability $\frac{1}{2}$ to each of the sequences $\ldots, 0, 0, 0, \ldots$ and $\ldots, 1, 1, 1, \ldots$.

7. Recall the continuous-time two-dimensional quadrant-growth model discussed in class, where the unit square with lower-left corner (i, j) $(i, j \in \mathbf{Z})$ is on at time 0 iff i or j is negative, and where, for $i, j \geq 0$, the associated square turns on at rate 1 once the squares with lower-left corners $(i-1, j)$ and $(i, j-1)$ are on. Guess (by computer simulation, heuristic arguments, or rigorous proof) the asymptotic shape for the set of turned-on squares in the first quadrant, scaled by $1/t$, as $t \to \infty$. (Hint: It is of the form $x^c + y^c = 1$ for some constant c.)