18.177: Problem Set #3 (due Thursday, April 11)

1. Let $X = \{0,1\}^{\mathbf{Z}}$ with $(\frac{1}{2},\frac{1}{2})$ i.i.d. measure. Define a random variable $f(x) = x_0 x_1 \in \{0,1\}$. Does the stochastic process $(\ldots, f \circ T^{-1}, f, f \circ T, \ldots)$ have full entropy (i.e., entropy $\log 2$)? What about the process determined by $g(x) = x_0 + x_1 \in \{0,1,2\}$?

2. Let T be a measure-preserving bijection from (X,μ) to itself whose inverse is also measure-preserving, and let P be a measurable finite partition of X.

(a) Must $H(P \vee TP \vee T^2P) = H(P \vee T^{-1}P \vee T^{-2}P)$?
(b) Must $H(P \vee TP \vee T^3P) = H(P \vee T^{-1}P \vee T^{-3}P)$?
(c) Show that $h(T,P) = h(T^{-1},P)$.

3. Show that $h(T^2) = 2h(T)$. (It can be shown more generally that if f is an integrable, positive-integer-valued measurable function on X, with (X,T,μ) ergodic, then the integral transformation T^f satisfies $h(T^f) = (\int f\, d\mu)h(T)$.) Note: If you prefer, you may assume that the dynamical system in question comes equipped with a generating partition P.

4. Show that $h_\mu(Q,T)$ and $h_\mu(T)$ are affine functions of μ on the space of T-invariant probability measures on $(X,\mathcal{F})$. Are they continuous functions of μ in the weak-* topology? Explain. (Note: You may assume the existence of a single partition P such that, for every T-invariant measure μ, P generates the full σ-algebra modulo sets of μ-measure 0; the weak-* topology in question is then the topology associated with the P-cylinder sets. However, the result is true more generally.)

5. Given two finite partitions P, P' of X (with (X,T,μ) ergodic) such that both P and P' have r components, indexed from 1 to r, define the distance between P and P' as $\mu(\bigcup_i P_i \triangle P'_i)$. (Note: This is a metric between *ordered* partitions, that is, partitions in which the components come equipped with some definite ordering, from "first" through "last".) Is $h(T,P)$ continuous relative to this metric on partitions P?

(more)

6. Given two finite μ-preserving transformation T, T' of X (with (X, T, μ) ergodic), define the distance between T and T' as $\mu(\{x : Tx \neq T'x\})$. Is $h(T, P)$ continuous relative to this metric on transformations T?

7. For n some fixed positive integer, there is a unique square-integrable real-valued random variable X taking its values in $\frac{1}{n}\mathbf{Z}$ that has (a) mean 0, (b) variance 1, and (c) as large an entropy $H(X)$ as is consistent with constraints (a) and (b). What is the limit of the distribution of X as $n \to \infty$ (so that X becomes increasingly continuous)?