18.177: Problem Set #2 (due March 21)

1. Let T be an ergodic measure-preserving invertible transformation of a non-atomic probability space $(X, \mathcal{F}, \mu)$.
(a) Given a measurable set A with $\mu_A > 0$, show that the induced transformation T_A is ergodic.
(b) Given a measurable function f with $\int f \, d\mu < \infty$, show that the integral transformation T_f is ergodic.

2. Let T be an ergodic measure-preserving invertible transformation of a non-atomic probability space $(X, \mathcal{F}, \mu)$. Show that there exists a set A of positive measure for which the return time $n_A(x) = \min\{n > 0 : T^n x \in A\}$ is unbounded (in the sense that $\{x : n_A(x) > k\}$ has positive measure for all k).

3. Let T be an ergodic measure-preserving invertible transformation of a non-atomic probability space $(X, \mathcal{F}, \mu)$, and let k be an arbitrary positive integer. Show that there exists a set A of positive measure such that for almost all x in A, the return time $n_A(x)$ is either k or $k + 1$.

4. Let $U : L^2(X, \mathcal{F}, \mu) \to L^2(X, \mathcal{F}, \mu)$ be the unitary operator associated with an ergodic measure-preserving invertible transformation on a non-atomic probability space $(X, \mathcal{F}, \mu)$. Show that every point on the unit circle is an *approximate eigenvalue* of U; i.e., given $\lambda \in \mathbf{C}$ with $|\lambda| = 1$ there exist functions $f_n \in L^2$ with $||f_n||_2 = 1$ for all n and $||Uf_n - \lambda f_n||_2 \to 0$.

5. Let T be an ergodic measure-preserving invertible transformation of a non-atomic probability space $(X, \mathcal{F}, \mu)$, and let T' be an ergodic measure-preserving invertible transformation of a non-atomic probability space $(X', \mathcal{F}', \mu')$. Show that for all $\epsilon > 0$ there exists a measure-preserving bijection $\Phi : X \to X'$ such that $T'(\Phi(x)) = \Phi(T(x))$ for all x outside a set of measure ϵ.

(more)

6. Imagine an "infinite binary odometer" whose states are mappings from $\mathbf{N}$ to $\{0,1\}$ (to be thought of in reverse order so that the zeroeth bit is on the right), and define the map T (defined almost everywhere) that changes the rightmost 0 into a 1 and changes all 1's to the right of the point into 0's. Let μ be the usual product measure on $\{0,1\}^{\mathbf{N}}$. Prove that T preserves the measure μ and that the system is ergodic. Determine all eigenvalues of the system. Does this dynamical system have discrete spectrum?

7. If α is the golden ratio and β is its square, then the sets $N(\alpha) = \{\lfloor k\alpha \rfloor : k \geq 1\}$ and $N(\beta) = \{\lfloor k\beta \rfloor : k \geq 1\}$ are disjoint. (Proof: For all $n \geq 1$, the fact that n/α and n/β are non-integers summing to n implies that $|N(\alpha) \cap (0,n]| + |N(\beta) \cap (0,n]| = \lfloor n/\alpha \rfloor + \lfloor n/\beta \rfloor = n-1$; since this is precisely 1 more than $n-2 = |N(\alpha) \cap (0,n-1]| + |N(\beta) \cap (0,n-1]|$, exactly one of the two sets $N(\alpha)$, $N(\beta)$ contains n.) Use ergodic theory to show that when α and β are independent in a suitable sense (which you must make precise), the sets $N(\alpha)$ and $N(\beta)$ intersect in a set of density $1/\alpha\beta$. (Hint: Note that $n \in N(\alpha)$ if and only if the fractional part of n/α lies between $1 - \frac{1}{\alpha}$ and 1, and similarly for β; look at the map of $[0,1) \times [0,1)$ that sends (x,y) to $(x + \frac{1}{\alpha}, y + \frac{1}{\beta})$ (mod 1).)

8. Let T be a measure-preserving invertible transformation of a non-atomic probability space $(X, \mathcal{F}, \mu)$. Assume that T^n is ergodic for all n, and that there is a constant c such that $\limsup_{n\to\infty} \mu(T^n A \cap B) \leq c\mu(A)\mu(B)$ for all $A, B \in \mathcal{F}$. Prove that T is strongly mixing. (Hint: First prove that T is weakly mixing. Then let m be a cluster point of the measures ν_n on $X \times X$ defined by

$$\int f(x,y)d\nu_n = \int f(x,T^n x)d\mu.$$

Show that $m = \mu \times \mu$.)