Walks on Walks

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Slides for this talk are on-line at

http://jamespropp.org/mathfest12a.pdf

Acknowledgments

This talk describes on-going work with Tom Roby.

Thanks also to Drew Armstrong, Karen Edwards, Bob Edwards, Svante Linusson, Richard Stanley, and Ben Young.

Michael La Croix created the awesome animations (and Tom Roby helped him adapt them for my talk and help me learn how to use them).

Walks

Given positive integers a, b, and $n = a + b$, there are $n!/a!b!$ ways to take a walk in $\mathbb{Z}^2$ from $(-a, a)$ to (b, b) consisting of a steps of type $(+1, -1)$, or **downsteps**, and b steps of type $(+1, +1)$, or **upsteps**.

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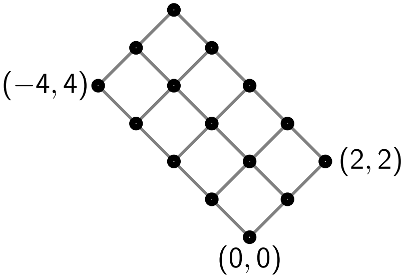

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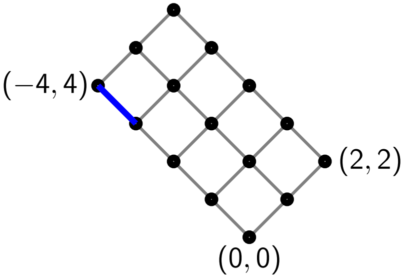

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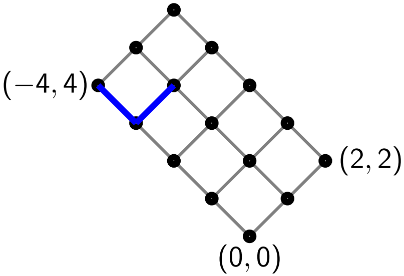

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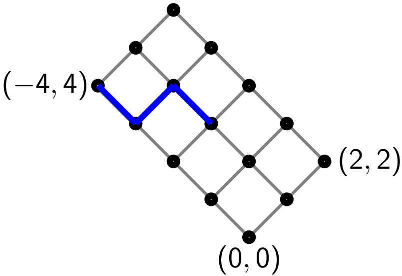

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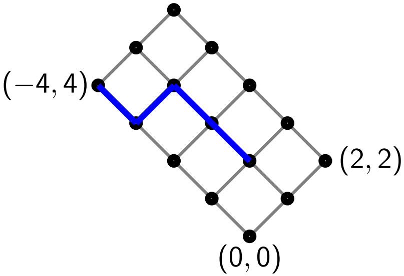

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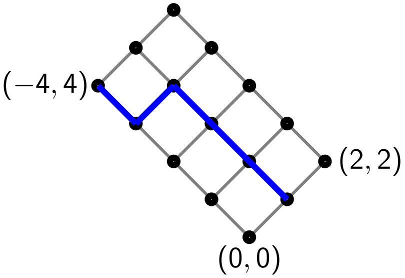

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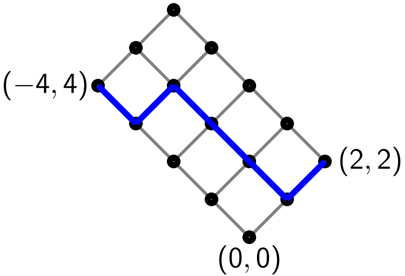
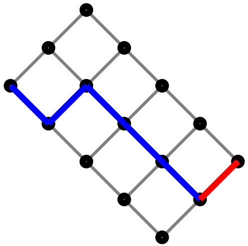


A walk on walks

Given a walk, or lattice path, P, from $(-a, a)$ to (b, b), we can do a cyclic shift of the $a + b$ steps, obtaining a new lattice path promotion(P) from $(-a, a)$ to (b, b).



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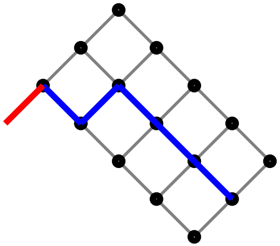

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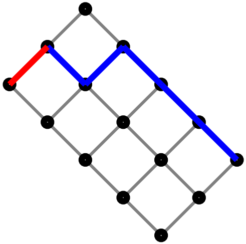


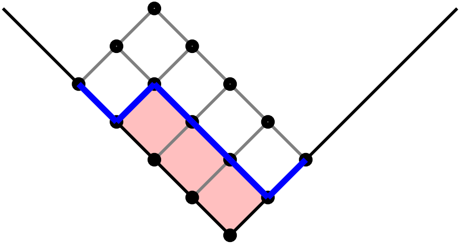
Promoting paths

If we apply promotion $n = a + b$ times, we get back the original lattice path. See `PromotionOrbit.pdf`.

So we have an action of the cyclic group $\mathbb{Z}/n\mathbb{Z}$ on the set of lattice paths from $(-a, a)$ to (b, b).

Paths enclose areas

For each path P, define $A(P)$ as the area bounded by P and the graph of $y = |x|$ (using tilted squares as units).



In this example $A(P) = 3$.

Averaging areas

Claim 0: The average of $A(P)$ over all paths P from $(-a, a)$ to (b, b) is $ab/2$.

Proof:

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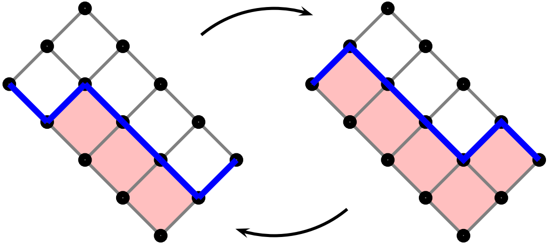

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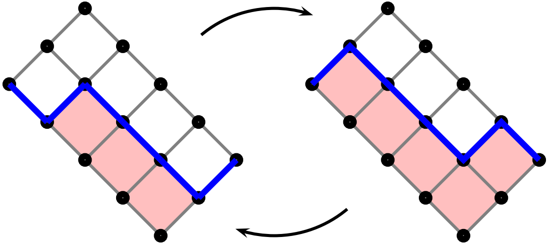

The average of $A(P)$ over each pair (or singleton) is $ab/2$.

Averaging over orbits

Claim 1: The average of $A(P)$ over all paths P within each promotion-orbit $\mathcal{O}$ is $ab/2$.

See `PromotionAverage.pdf`.

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Before I show you the proof, let's consider another fact with the same flavor.

Another walk on walks

Here's another operation on paths, called rowmotion:

1) Everywhere the path P contains a downstep followed by an upstep, mark the beginning of the downstep and the end of the upstep by a red dot.

2) Also mark the endpoints of the path by red dots.

3) Take each path-segment bounded by red dots and rotate it in place by 180 degrees.

The new path is rowmotion(P).

See `RowmotionOrbit.pdf`.

Un-walking the walk

Before we do the rotation, the dots demarcate the places where we see a downstep followed by an upstep;
after we do the rotation, the dots demarcate the places where we see an upstep followed by a downstep.

Remember this.

This observation implies that rowmotion is *reversible*.

Row, row, row, ...

It can be shown (Fon-Der-Flaass, 1993, with subsequent clarifications by Stanley and Armstrong) that if we apply rowmotion $n = a + b$ times, we get back the original lattice path.

So we have a **different** action of the cyclic group $\mathbb{Z}/n\mathbb{Z}$ on the set of lattice paths from $(-a, a)$ to (b, b).

Claim 2: The average of $A(P)$ over all paths P within each rowmotion orbit $\mathcal{O}$ is $ab/2$.

See `RowmotionAverage.pdf`.

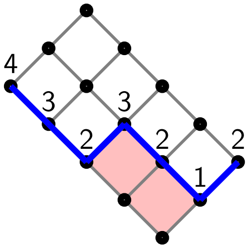
Averaging the paths themselves

In fact, Claims 1 and 2 follow from stronger claims about orbit-averages, where we average not the values of $A(P)$ but the P's themselves, viewed as real-valued functions from $[-a, b]$ to $\mathbb{R}$.

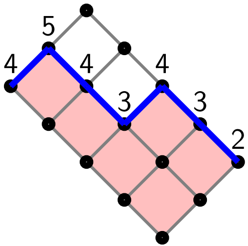
Claim $1'$: The average of $P(\cdot)$ over each promotion-orbit $\mathcal{O}$ is the linear function $L(\cdot)$ whose graph goes through $(-a, a)$ and (b, b).

Claim $2'$: The average of $P(\cdot)$ over each rowmotion-orbit $\mathcal{O}$ is the linear function $L(\cdot)$ whose graph goes through $(-a, a)$ and (b, b).

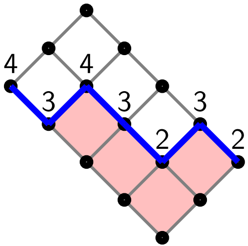
Averaging paths and averaging areas: an example



$Area = 2$

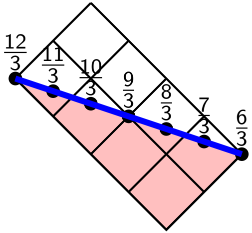


$Area = 6$



$Area = 4$

Average:



$Area = 4$

Reducing Claims 1′ and 2′ to Claims 1 and 2

Let $\mathcal{O}$ denote an orbit (for now, it doesn't matter whether it's a promotion orbit or a rowmotion orbit).

Let $\overline{P}_{\mathcal{O}}(\cdot)$ denote the average of the functions $P(\cdot)$ within $\mathcal{O}$.

On the one hand, the area bounded by $\overline{P}_{\mathcal{O}}(\cdot)$ equals the average of $A(P)$ within $\mathcal{O}$.

On the other hand, Claims 1′ and 2′ imply that the area bounded by $\overline{P}_{\mathcal{O}}(\cdot)$ equals the area bounded by $L(\cdot)$, which is $ab/2$.

So Claims 1′ and 2′ imply Claims 1 and 2.

Averaging paths and averaging values

Each function $P : [-a, b] \to \mathbb{R}$ is linear on each interval $[n, n+1]$ (with n an integer), so $\overline{P}_{\mathcal{O}}$ is too.

So to prove Claim $1'$ or $2'$, it's enough to prove that for every integer n in $[-a, b]$, $\overline{P}_{\mathcal{O}}(n) = L(n)$.

We know that it's true for $n = -a$ and $n = b$ (since $P(-a) = a = L(a)$ and $P(b) = b = L(b)$ for all paths P); what about values of n in between?

Averaging values and averaging differences of values

We know that the values of $L(\cdot)$ at $-a, -a+1, \ldots, b$ form an arithmetic progression with first term a and last term b.

We also know that $\overline{P}_{\mathcal{O}}(-a) = a$ and $\overline{P}_{\mathcal{O}}(b) = b$.

To prove that $\overline{P}_{\mathcal{O}}(n) = L(n)$ for all n, it's enough to show that $\overline{P}_{\mathcal{O}}(-a)$, $\overline{P}_{\mathcal{O}}(-a+1)$, ..., $\overline{P}_{\mathcal{O}}(b)$ form an arithmetic progression.

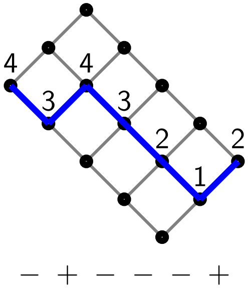
We'll do this by showing that the difference between each term and the next doesn't change as you move through the sequence $\overline{P}_{\mathcal{O}}(-a)$, $\overline{P}_{\mathcal{O}}(-a+1)$, ..., $\overline{P}_{\mathcal{O}}(b)$.

Difference sequences

Represent each P by its **difference sequence**

$$(P(n) - P(n-1) : \ a+1 \leq n \leq b),$$

consisting of a -1's and b $+1$'s, which we'll sometimes write as "$+$" and "$-$".
These represent the slopes of the path-segments.



Difference arrays

For simplicity, assume $\#(\mathcal{O}) = n$.
Create an n-by-n array, in which the kth row is the difference sequence for the kth element of an orbit $\mathcal{O}$.
Call it the **difference array** for the orbit.

The average of the $+1$'s and -1's in the nth column of the difference array is equal to the average of $P(n) - P(n-1)$ as P varies over $\mathcal{O}$, which is just $\overline{P}_{\mathcal{O}}(n) - \overline{P}_{\mathcal{O}}(n-1)$.

So all we need to do is to show that each column has the same average as the next.

Proof of Claim 1′ (promotion)

$$\begin{array}{cccccc}
- & - & - & + & - & + \\
+ & - & - & - & + & - \\
- & + & - & - & - & + \\
+ & - & + & - & - & - \\
- & + & - & + & - & - \\
- & - & + & - & + & -
\end{array}$$

Since each row of the difference array is the rightward shift of the row above it, each column is the downward shift of the column to its left.

In particular, each column has the same number of $+$'s and $-$'s as the next, so each column has the same average as the next, QED.

Proof of Claim 2′ (rowmotion)

$$\begin{array}{cccccc} - & - & - & - & + & + \\ - & - & - & + & - & + \\ - & - & + & - & + & - \\ - & + & - & + & - & - \\ + & - & + & - & - & - \\ + & + & - & - & - & - \end{array}$$

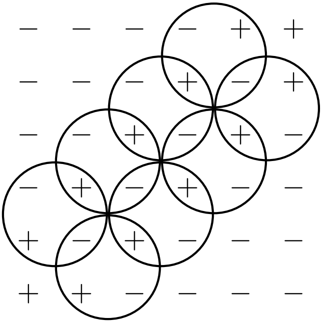
Proof of Claim 2′ (rowmotion)

$$\begin{array}{cccccc}
- & - & - & - & + & + \\
- & - & - & + & - & + \\
- & - & + & - & + & - \\
- & + & - & + & - & - \\
+ & - & + & - & - & - \\
+ & + & - & - & - & -
\end{array}$$

Remember that a path P has a downstep followed by an upstep precisely where rowmotion(P) has an upstep followed by a downstep.

So a row in the difference array has a $-$ followed by a $+$ if and only if the following row (with wraparound) has a $+$ followed by a $-$.

Proof of Claim 2′ (concluded)



Hence if we look at two consecutive columns, the places where they differ pair up: places with a $-$ to the left of a $+$ are paired with places with a $+$ to the left of a $-$.
So each column has the same number of $+$'s and $-$'s as the next, so each column has the same average as the next, QED.

What is the context for results like this?

For many cyclic actions τ on a finite set S of combinatorial objects, and for many natural statistics ϕ on S, the average of ϕ over each τ-orbit in S is the same as the average of ϕ over the whole set S.

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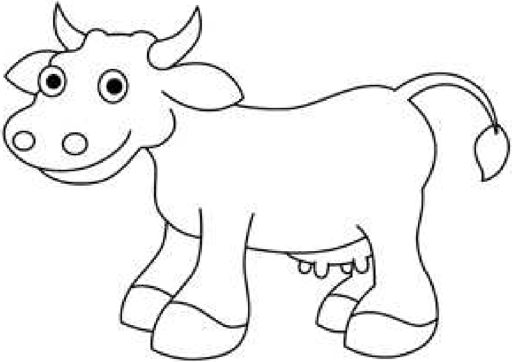
For many cyclic actions τ on a finite set S of combinatorial objects, and for many natural statistics ϕ on S, the average of ϕ over each τ-orbit in S is the same as the average of ϕ over the whole set S.

We say that (S, τ, ϕ) exhibits combinatorial ergodicity, or the CAAO (Constant Averages Along Orbits) property.



Linearity

A key tool in proving that the CAAO property holds for a specific triple is the fact that, for fixed S and fixed $\phi : S \to S$, the set of ϕ's for which (S, τ, ϕ) has the CAAO property is a vector space: linear combinations of ϕ's with the CAAO property have the CAAO property too.

In our promotion and rowmotion proofs, we used linearity to deduce the CAAO property for the area $A(P)$ from the CAAO property for the differences $P(-a+1) - P(-a), \ldots, P(b) - P(b-1)$.

Riddle

A Russian colleague of mine was complaining about the students in one of his classes, and it sounded like he said:

"I am in middle of lecture, students are walking in, other students are walking out; is cows!"

What did he really say?

The future

Tom Roby and I have found numerous examples of triples with the CAAO property; stay tuned for a preprint coming soon to a home-page near you.

If you're in a boring talk:
Take your favorite example of a bijection from a set of combinatorial objects to itself, and your favorite statistic on such objects, and compute the average of the statistic on each orbit of the action.
You may find that the statistic exhibits combinatorial ergodicity.

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