Spanning trees in high-dimensional grids: proposed by James Propp

Robin Pemantle (answering a question raised by Russell Lyons) has shown that if one takes a uniform spanning tree of a d-dimensional grid-graph, with n vertices on a side, the resulting distribution has a weak limit as n goes to infinity, for each fixed d. In every case, this limit-measure is supported on the set of "essential spanning forests" in $\mathbf{Z}^d$; that is, the set of spanning forests of $\mathbf{Z}^d$ in which each component is infinite. However, only for $d \leq 4$ does one actually get a distribution that is supported on the set of spanning trees of $\mathbf{Z}^d$.

What is one to do about this situation?

One attitude to take is that there is nothing to be done, and that there is no canonical spanning tree process in high-dimensional grids. How could we make such a negative statement precise?

One result in this direction is the observation (conveyed to me by Lyons) that for $d > 4$, no translation-invariant tree-valued process can stochastically dominate the Pemantle process (that is, a translation-invariant tree-valued process cannot be obtained by starting from a forest given by the Pemantle measure and then adding some edges in a translation-invariant way); one proves this by noting that both subgraphs of $\mathbf{Z}^d$ must contain half of the edges of the graph.

Here is another negative result along these lines: Recall that one property that singles out the Pemantle measure on essential spanning forests in $\mathbf{Z}^d$ ($d \geq 2$) is entropy-maximization. That is, every translation-invariant measure supported on the set of spanning trees of $\mathbf{Z}^d$ has an entropy, and Robert Burton and Robin Pemantle showed that the Pemantle measure is the only one that has as its entropy the supremum of all the entropies of all such measures. However, for $d > 4$ this measure is not a "spanning tree measure" (that is, it is not supported on the set of spanning trees in $\mathbf{Z}^d$). Hence for any $d > 4$ the aforementioned supremum is not achieved within the set of tree-measures, despite the fact (as communicated to me by Lyons) that one can get arbitrarily close. (Note that the variational principle, normally invoked in situations where one wants to prove that an entropy-supremum is achieved, does not apply here, because the set of measures in question is not compact.)

The other attitude to take is that there *is* a natural translation-invariant model on spanning trees of $\mathbf{Z}^d$, to be found in some other way.

Perhaps we need to take Pemantle's essential spanning forests and add more edges in a translation-invariant way, removing other edges as we go when necessary. Let us therefore introduce independent rate 1 Poisson processes associated with the respective edges of the grid-graph $\mathbf{Z}^d$, and decree that when a timer goes off, the associated edge joins the forest — with the proviso that if this change creates a cycle, then one of the edges in that cycle, chosen uniformly at random, should be deleted from the forest. One encouraging feature of this scheme is that if it is implemented on a finite graph, it has the uniform measure on spanning trees as its unique stationary distribution (the same is not the case for the seemingly more natural algorithm in which edges are added at random except when this would create a cycle). However, it is not clear that this stochastic process is well-defined; even though the edges that are added in the time-interval $(t, t + \Delta t)$ are spread far apart, they give rise to cycles of unbounded length, and if these cycles percolate there may be no way to say "what is supposed to happen". It is not clear to me whether things get better or worse in this regard as the dimension d increases. (I should mention that David Aldous seems to be the first to have studied this process on finite graphs, and that Oded Schramm seems to have independently considered the extension to infinite graphs.)

There remains the very apt question "How do we obtain a measure from this process?" The process certainly does not converge to stationarity in finite time. Clearly what we must do is take the limit of the behavior at time t as $t \to \infty$. But it is possible that this limit measure will not be supported on the set of spanning trees.