EDGE EFFECTS ON LOCAL STATISTICS IN LATTICE DIMERS:
A STUDY OF THE AZTEC DIAMOND (FINITE CASE)

BY

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THESIS

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1. Introduction

A *tiling* of a checkerboard with dominoes is a way of putting dominoes on the board so that no square of the board is uncovered and no two dominoes overlap. Given a local pattern (see figure 1 for examples), a location in the board, and the shape and size of the board, how many tilings of the board have the given pattern at the given location? (Alternatively, we can substitute "bond" for "domino" and "particle" for "square", and ask for the probability of local patterns in a system of particles each of which bonds with exactly one of its neighbors.)

Suppose that the squares of the board are very small compared to the board itself. For some board shapes, the probability of finding a pattern at a given location will be the same for almost all locations. This is the case for the square board. (See figures 3 to 5, where tiles are colored according to their direction and parity for the sake of clarity; see figure 6 for the coloring scheme.) There are some boards, however, for which the probability does depend on the location. Consider, for example, the *Aztec diamond*, that is, the board whose boundary is a square tilted 45 degrees (figures 7 and 8). In random tilings of the Aztec diamond, we usually find brick-wall patterns outside the inscribed circles, and more complicated behavior inside the circle. (See figures 9 to 13.)

The probabilities of local patterns in a rectangular board were computed recently [6]. Until now, there was no other board for which the probabilities of all local patterns were known. Many experiments and some important partial results [1] had shown that, as already stated, the probabilities of patterns in the Aztec diamond depend on location. This qualitative difference between the Aztec diamond and the rectangular board made the former as worthy of analysis as the latter. The main result of this work is an expression for the probability of any local pattern in a random tiling of the Aztec diamond. The expression is a determinant of size proportional to the number of squares in the pattern, just like Kenyon's expression [6] for the probabilities in the rectangural board,

Main Result 1. *The probability of a pattern covering white squares $v_1, v_2, \cdots v_k$ and black squares $w_1, w_2, \cdots w_k$ of an Aztec diamond of order n is equal to the absolute value of*

$$\left| c(v_i, w_j) \right|_{i,j=1,2,\cdots k}.$$

The coupling function $c(v, w)$ *at white square v and black square w is*

$$2^{-n} \sum_{j=0}^{x_i - 1} \mathrm{kr}(j, n, y_i)\, \mathrm{kr}(y\prime_i - 1, n-1, n-(j + x\prime_i - x_i))$$

for $x\prime_i > x_i$ and

$$-2^{-n} \sum_{j=x_i}^{n} \mathrm{kr}(j, n, y_i)\, \mathrm{kr}(y\prime_i - 1, n-1, n-(j + x\prime_i - x_i))$$

for $x\prime_i \le x_i$, where (x_i, y_i) and $(x\prime_i, y\prime_i)$ are the coordinates of v and w, respectively, in the coordinate system in figure 17, and the Krawtchouk polynomial $\mathrm{kr}(a, b, c)$ *is the coefficient of x^a in $(1-x)^b \cdot (1+x)^c$.*

Our line of attack is as follows.

1. Reduce the problem of finding probabilities of patterns to an enumerative problem;

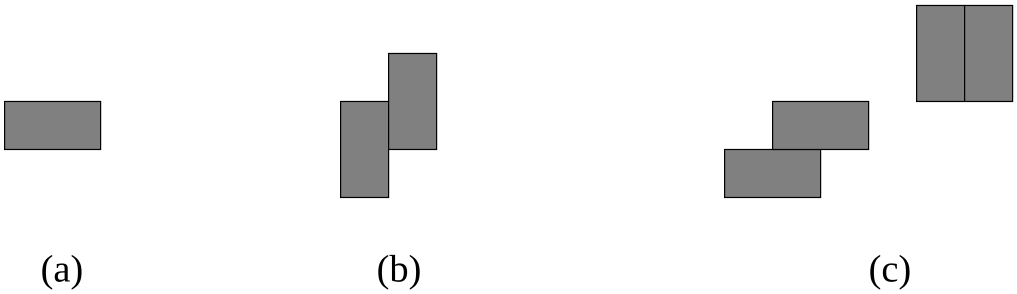

FIGURE 1. A few examples of local patterns

FIGURE 2. These two patterns have the same probability of being found in a random pattern at any given place

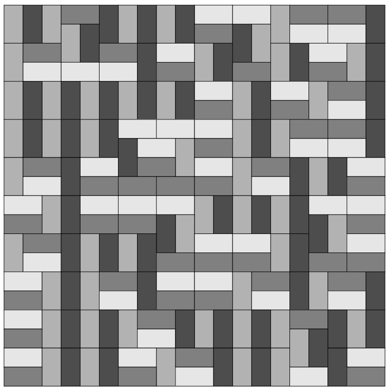

FIGURE 3. Random tiling of square of side 20

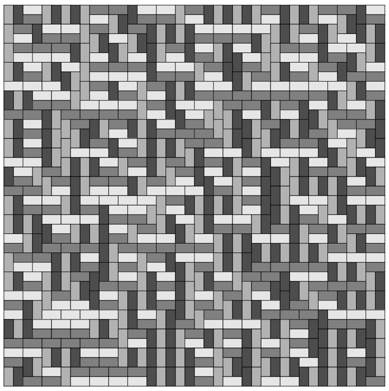

FIGURE 4. Random tiling of square of side 40

2. Reduce the enumerative problem to a simpler one involving Aztec diamonds with two holes rather than arbitrary even-area holes;
3. Compute the weighted number of tilings of an Aztec diamond with two holes.

The first two steps involve known techniques, and were already considered to be a plausible strategy by other researchers. The third step is new.

Henry Cohn is currently analyzing the case of the board with infinitely small squares by approximating the sum of Krawtchouk polynomials in our main result as an integral for $n \to \infty$. His results will be presented in a later, joint version of this paper.

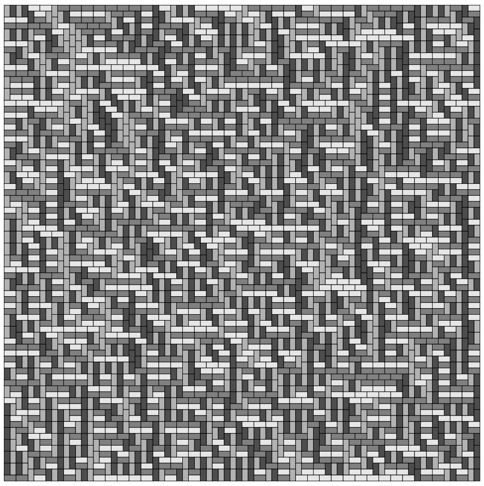

Figure 5. Random tiling of square of side 80

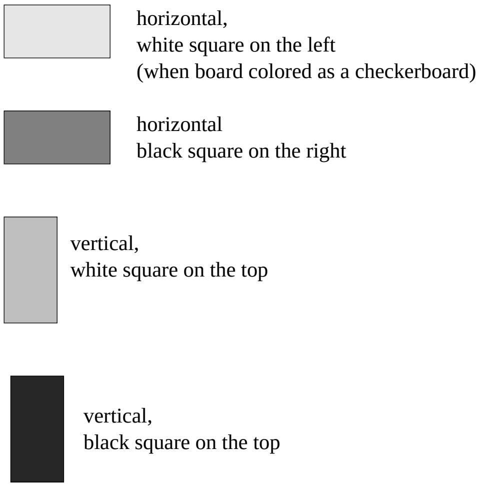

Figure 6. Shading chart

2. The Kasteleyn Matrix

What is the probability of finding a pattern in a tiling of a given board? It is equal, by definition, to the number of tilings of the board with the given pattern, divided by the total number of tilings of the board. Clearly, the number of tilings of the board with the given pattern depends only on the squares occupied by the pattern, and not by how they are tiled by the pattern: there is a one-to-one correspondence between tilings with the given pattern, and tilings of the board with the squares covered by the pattern removed. (See figure 2.) Thus, what we want to know is the number of tilings of the board with the squares covered by a pattern removed, divided by the number of tilings of the board.

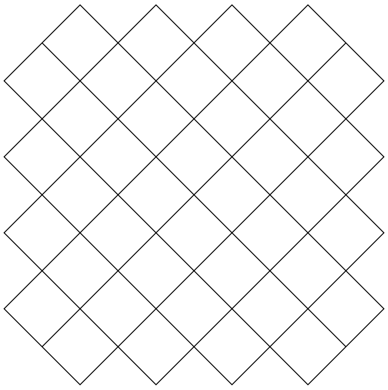

FIGURE 7. Aztec diamond of order 4, as a board

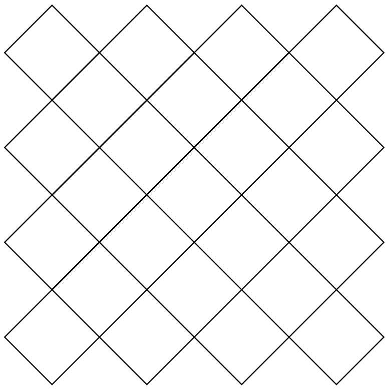

FIGURE 8. Aztec diamond of order 4, as a graph

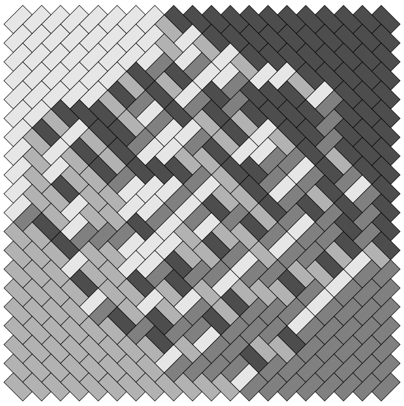

FIGURE 9. Random tiling of Aztec diamond of order (side) 20

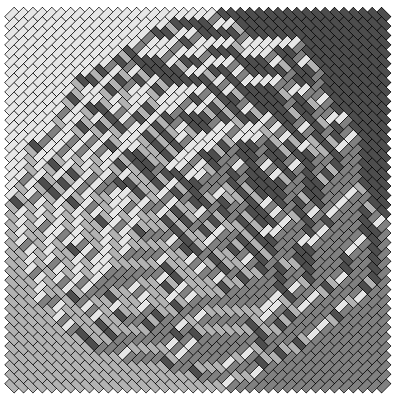

FIGURE 10. Random tiling of Aztec diamond of order 40

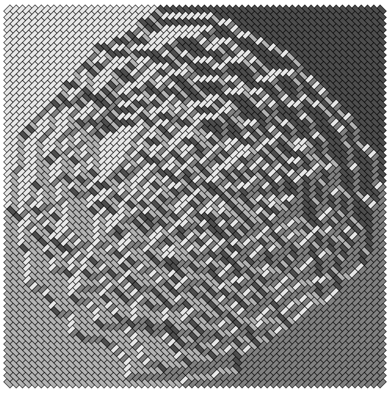

FIGURE 11. Random tiling of Aztec diamond of order 60

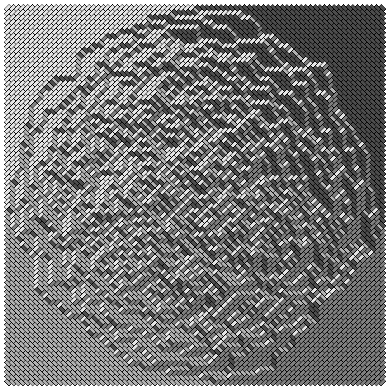

FIGURE 12. Random tiling of Aztec diamond of order 80

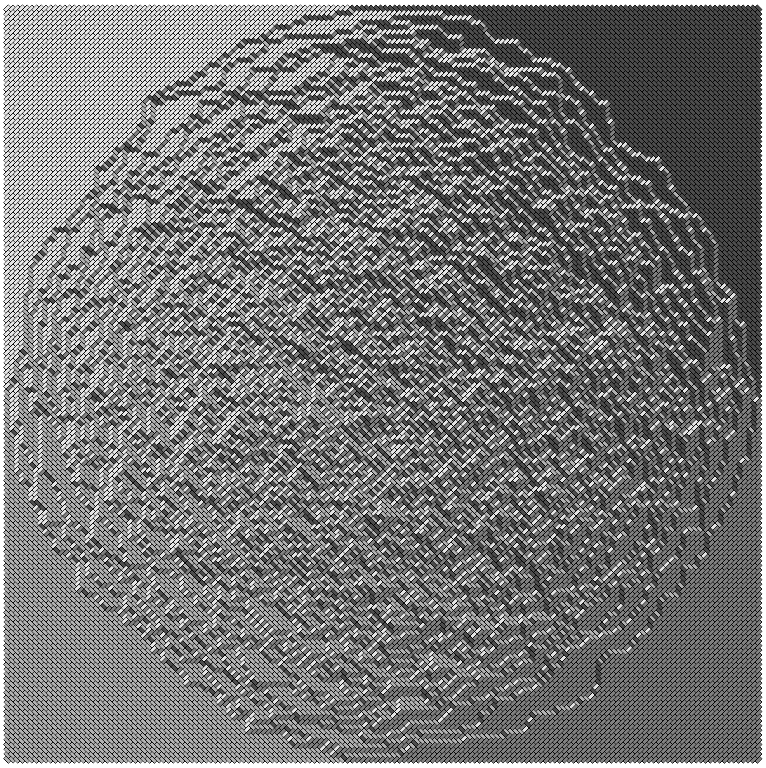

FIGURE 13. Random tiling of Aztec diamond of order 160

Kasteleyn [5] showed that the number of tilings of a board can be expressed as the absolute value of a determinant with half as many rows as there are squares in the board. Thus, in a sense, our problem is already solved: since determinants can be computed in time polynomial on the number of rows, we can compute the probability of any pattern in a board in time polynomial on the size of the board. Unfortunately, there are two disadvantages to this approach. The first one is that, if the Aztec diamond in question has side n, we will have to compute a determinant of side $\frac{n^2}{2}$, and that demands a considerable amount of time and space. Even more seriously, a sequence of determinants of varying size is very hard to analyze asymptotically. We would like to analyze such a sequence in order to determine the probability of a pattern near a point for Aztec diamonds of very high order, that is, of very fine "grain" (see figure 13). Thus, a Kasteleyn determinant is not good enough. A determinant whose size depended only on the size of the pattern, and not on the size of the board, would be much easier to manipulate.

In this section, we will prove that the number of tilings of a board equals the absolute value of the determinant of its Kasteleyn matrix, and then show how this implies that the probability of a pattern in a random tiling of a board is equal to a minor of the inverse of the Kasteleyn matrix of the board. This minor has side proportional to the number of squares in the pattern. Finally, we will show that the problem of finding an entry in the inverse of a Kasteleyn matrix can be reduced to an enumerative problem concerning an Aztec diamond with one black hole and one white hole.

We will henceforth refer, not to boards, but to their dual graphs, which can be seen as having a vertex at the center of every square and an edge perpendicular to every edge between two squares. For example, the Aztec diamond will mean for us the object in figure 8 and not the object in figure 7. This convention will simplify graph-theoretical arguments considerably.

The results in this section are in part a modern formulation of Kasteleyn's work, and in part a codification of local folklore, as crafted and passed down by R. Kenyon, J. Propp, D. Wilson and others.

2.1. **The Kasteleyn-Wilson matrix.** Consider a finite subgraph G of the infinite square lattice with as many white as black vertices, where the infinite square lattice is colored as a checkerboard. Let $(v_1, \cdots, v_n)$ be its white vertices and $(w_1, \cdots, w_n)$ its black vertices. (Any ordering from 1 to n can be chosen.) The Kasteleyn-Wilson matrix

$$K((v_1, \cdots, v_n), (w_1, \cdots, w_n))$$

(or $K(G)$, by abuse of language) is defined to be $|a_{i,j}|_1^n$, where $a_{i,j}$ is

- 0, if $\{v_i, w_j\}$ is not an edge of G;
- 1, if $\{v_i, w_j\}$ is a horizontal edge of G;
- $(-1)^k$, if $\{v_i, w_j\}$ is a vertical edge going from row l to row $l+1$ and there are k vertices in row l to the left of the edge.

We will show that the number of perfect matchings of G is equal to the absolute value of the determinant of $K(G)$. As could be exprected, the Kasteleyn-Wilson matrix is only one of several Kasteleyn matrices $K(G)$, that is, determinants whose absolute values are equal to the perfect matchings of G. We will prove the enumerative property of the Kasteleyn-Wilson matrix because it is true for any subgraph G of the infinite square lattice, and not only for the Aztec diamond. Fortunately, the proof can be applied to other Kasteleyn matrices with minimal cases. In fact, in the last subsection of this section, and in following sections, we will use the following convention, which is more convenient for our purposes than Wilson's:

- 0, if $\{v_i, w_j\}$ is not an edge of G;
- 1, if $\{v_i, w_j\}$ is a horizontal edge of G;
- $(-1)^k$, if $\{v_i, w_j\}$ is a vertical edge going from row l to row $l+1$ and there are k vertical edges from row l to row $l+1$ to the left of the edge.

This convention is valid for the Aztec diamond and for any other subgraph G of the infinite square lattice such that any lattice vertex inside any loop in G is also a vertex of G. Thus, only the material in the following two subsections is valid for any subgraph of the infinite square lattice. The rest of this work is specific to the Aztec diamond, although the techniques used are applicable to other boards.

2.2. **Why does K give us the number of perfect matchings?** We want to show that the number of perfect matchings of a subgraph G of the infinite square lattice is equal to the absolute value of the determinant of $K(G)$.

We can express $\det(K)$ as

$$\sum_{\pi \in P(\{1,2,\ldots,n\})} \operatorname{sgn}(\pi) \prod_{i=1}^{n} a_{i,\pi(i)}, \tag{1}$$

where $P(\{1,2,...,n\})$ is the set of all permutations of $\{1,2,\cdots n\}$. Define a map f from the set of all perfect matchings of G to $P(\{1,2,...,n\})$ as follows. Any perfect matching can be expressed in the form

$$\{\{v_1, w_{k(1)}\}, \{v_2, w_{k(2)}\}, \cdots, \{v_n, w_{k(n)}\}\}, \tag{2}$$

where k is a map from $\{1,2,\cdots n\}$ to itself. The map f takes $\{\{v_1, w_{k(1)}\}, \cdots, \{v_n, w_{k(n)}\}\}$ to k. It is clear that k is a permutation; otherwise there would be unpaired vertices, as well as vertices belonging to more than one pair. Thus, f is well defined. Moreover,

1. f is injective: If two matchings had the same map k, they would be the same matching.
2. every element k of the image of f satisfies $(\prod_{i=1}^{n} a_{i,k(i)}) \neq 0$: if $v_i, w_{k(i)}$ is an edge, then $a_{i,k(i)}$ must be non-zero.
3. if a permutation π of $1,2,...,n$ satisfies $\prod_{i=1}^{n} a_{i,\pi(i)} \neq 0$, then, for every $1 \leq i \leq n$, $(v_i, w_{\pi(i)})$ is a valid edge. Moreover, for $i_1 \neq i_2$, $\pi(i_1) \neq \pi(i_2)$, and thus $(v_{i_1}, w_{\pi(i_1)})$ and $(v_{i_2}, w_{\pi(i_2)})$ do not have any vertices in common. Therefore
$$\{\{v_1, w_{k(1)}\}, \{v_2, w_{k(2)}\}, \cdots, \{v_n, w_{k(n)}\}\} \tag{3}$$
is a perfect matching.

Hence f is a one-to-one and onto map from the set of all perfect matchings of G to the set of all permutations k of $1,2,...,n$ satisfying $\prod_{i=1}^{n} a_{i,k(i)} \neq 0$. Therefore there are as many perfect matchings as there are non-zero terms in

$$\det(K) = \sum_{\pi \in P(\{1,2,\cdots,n\})} \operatorname{sgn}(\pi) \prod_{i=1}^{n} a_{i,\pi(i)}. \tag{4}$$

Every non-zero term is equal to either 1 or -1. To prove that the absolute value of $\det(K)$ equals the number of perfect matchings, we have to show that all non-zero terms have the same sign.

Let $M = \{\{v_i, w_{\pi(i)}\}\}_{i=1}^n$ and $M\prime = \{\{v_i, w_{\pi\prime(i)}\}\}_{i=1}^n$ be two perfect matchings of G. By the definition of perfect matching, every vertex of G is in one edge of M and in one edge of $M\prime$. It follows that every vertex of G is either in two edges or in no edges of $(M \cup M\prime) - (M \cap M\prime)$. Therefore $(M \cup M\prime) - (M \cap M\prime)$ consists entirely of loops, that is, it is a collection of disjoint sets of the form

$$\{\{v_{i_1}, w_{j_1}\}, \{w_{j_1}, v_{i_2}\}, \{v_{i_2}, w_{j_2}\}, \cdots \{v_{i_m}, w_{j_m}\}, \{w_{j_m}, v_{i_1}\}\}. \tag{5}$$

(See figure 14.) If a vertex is in two edges of $(M \cup M\prime) - (M \cap M\prime)$, one of these two edges must be in M, and the other one in $M\prime$. We can assume without loss of generality that $\{v_{i_1}, w_{j_1}\}$ is in M, and hence $\{w_{j_1}, v_{i_2}\}$ is in $M\prime$, $\{v_{i_2}, w_{j_2}\}$ is in M, and so on. Then, on one hand, $j_l = \pi(i_l)$ for $1 \leq l \leq m$, and, on the other hand, $j_l = \pi\prime(i_{l+1})$ for $1 \leq l \leq m-1$, $j_m = \pi\prime(i_1)$. Hence $i_2 = ((\pi\prime)^{-1} \circ \pi)(i_1)$, $i_3 = ((\pi\prime)^{-1} \circ \pi)(i_2)$, $\ldots$ $i_1 = ((\pi\prime)^{-1} \circ \pi)(i_m))$. Thus every loop in $(M \cup M\prime) - (M \cap M\prime)$ induces a cycle in $(\pi\prime)^{-1} \circ \pi$. It is easy to see that, conversely, for every cycle in $(\pi\prime)^{-1} \circ \pi$ there is a loop in $(M \cup M\prime) - (M \cap M\prime)$ that induces it. Because the sign of a permutation is equal to the product over all its cycles of (-1) to the power of the length of the cycle minus one, and because a cycle has length equal to half the number of edges of the loop inducing it, we have

$$\mathrm{sgn}((\pi\prime)^{-1} \circ \pi) = \prod_{\ell \in L} (-1)^{\mathrm{len}(\ell)/2-1}, \tag{6}$$

where L is the set of all loops of $(M \cup M\prime) - (M \cap M\prime)$ and $\mathrm{len}(\ell)$ is the number of edges in loop ℓ. From this equation, from

$$\mathrm{sgn}((\pi\prime)^{-1} \circ \pi) = \frac{\mathrm{sgn}(\pi)}{\mathrm{sgn}(\pi\prime)}, \tag{7}$$

and from the fact that all $a_{i,j}$ are 1 or (-1), it follows that the result we want to prove in this section, namely,

$$\mathrm{sgn}(\pi) \prod_{i=1}^n a_{i,\pi(i)} = \mathrm{sgn}(\pi\prime) \prod_{i=1}^n a_{i,\pi\prime(i)}, \tag{8}$$

is equivalent to

$$\prod_{\ell \in L} (-1)^{\mathrm{len}(\ell)/2-1} = \prod_{i=1}^n a_{i,\pi(i)} \cdot \prod_{i=1}^n a_{i,\pi\prime(i)}. \tag{9}$$

Now,

$$\prod_{i=1}^n a_{i,\pi(i)} \cdot \prod_{i=1}^n a_{i,\pi\prime(i)} = \prod_{\ell \in L} (\prod_{i \in I(\ell)} a_{i,\pi(i)} \cdot \prod_{i \in I(\ell)} a_{i,\pi\prime(i)}) \tag{10}$$

, where $I(\ell)$ is the set of indices i of all white vertices in loop ℓ. Therefore it is enough for us to prove that

$$(-1)^{\mathrm{len}(\ell)/2-1} = \prod_{i \in I(\ell)} a_{i,\pi(i)} \cdot \prod_{i \in I(\ell)} a_{i,\pi\prime(i)} \tag{11}$$

for every loop ℓ in $(M \cup M\prime) - (M \cap M\prime)$.

$\prod_{i \in I(\ell)} a_{i,\pi(i)} \cdot \prod_{i \in I(\ell)} a_{i,\pi\prime(i)}$ is the product $a_{i,j}$ over all $1 \leq i, j \leq n$ such that $\{v_i, w_j\}$ is an edge of loop ℓ. Since $a_{i,j} = 1$ for $\{v_i, w_j\}$ horizontal, we can restrict the product to $1 \leq i, j \leq n$ such that $\{v_i, w_j\}$ is a vertical edge of loop ℓ. What is, specifically, the product of $a_{i,j}$ over all $1 \leq i, j \leq n$ such that $\{v_i, w_j\}$ is a vertical edge of loop ℓ having

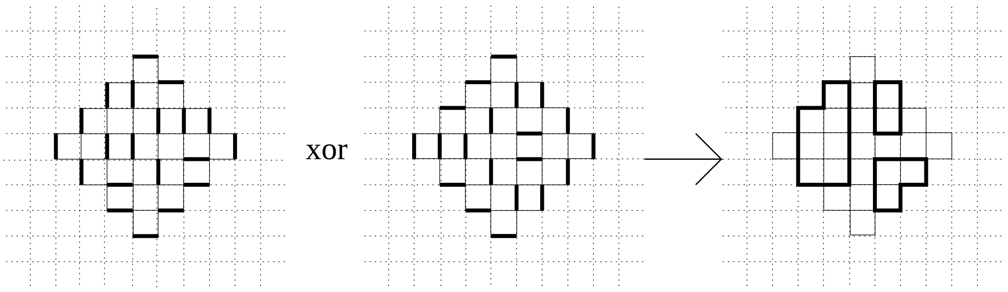

FIGURE 14. The union of two perfect matchings of the Aztec diamond, minus their intersection

a vertex on row y_0 and another in row $y_0 + 1$, where y_0 is a given? Let $\{(x_1, y_0), (x_1, y_0 + 1)\}, \{(x_2, y_0), (x_2, y_0 + 1)\}, \cdots \{(x_m, y_0), (x_m, y_0 + 1)\}$ be all such edges. (We are referring vertices by their Cartesian coordinates.) Since the horizontal line $\{(t, y + \frac{1}{2}) : t \in (-\infty, \infty)\}$ crosses the loop an even number of times, m must be even. Let $m = 2 \cdot m_0$. Since all $a_{i,j}$ are 1 or (-1), we have

$$\prod_{j=1}^{m} a_{J_0(x_j, y_0), J_1(x_j, y_0+1)} = \prod_{j=1}^{m_0} \frac{a_{J_0(x_{2j}, y_0), J_1(x_{2j}, y_0+1)}}{a_{J_0(x_{2j-1}, y_0), J_1(x_{2j-1}, y_0+1)}}, \tag{12}$$

where $J_0(z, w)$ is the index of the white vertex of coordinates (z, w), and $J_1(z, w)$ is the index of the black vertex of coordinates (z, w). Since $a_{J_0(x_i, y_0), J_1(x_i, y_0+1)}$ is equal to the number of vertices of G on row y_0 and to the left of x_i, and $a_{J_0(x_{i+1}, y_0), J_1(x_{i+1}, y_0+1)}$ is equal to the number of vertices of G on row y_0 and to the left of x_{i+1},

$$\frac{a_{J_0(x_{i+1}, y_0), J_1(x_{i+1}, y_0+1)}}{a_{J_0(x_i, y_0), J_1(x_i, y_0+1)}} \tag{13}$$

is equal to the number of vertices of G on row y_0 and to the left of x_{i+1} but not of x_i. This is the same as the number of vertices of the infinite square grid on row y_0 and to the left of x_{i+1} but not of x_i, minus the number of vertices in the grid but not in G, on row y_0 and to the left of x_{i+1} but not of x_i. The number of vertices of the grid to the left of x_{i+1} but not of x_i is equal to the number of squares of the grid contained between the edges $((x_i, y), (x_i, y + 1))$ and $((x_{i+1}, y), (x_{i+1}, y + 1))$.

It is clear that, when i is even, the squares between the edges $((x_i, y), (x_i, y + 1))$ and $((x_{i+1}, y), (x_{i+1}, y + 1))$ are in the interior of the loop, as are the vertices on row y of the grid which do not belong to G and which are the left of x_{i+1} but not of x_i. (Since any vertex on the loop, and, specifically, x_i and x_{i+1}, must be in G, we can hencefort refer to these vertices as "the vertices on row y of the grid which do not belong to G and which lie between x_i and x_{i+1}".) Conversely, every square between rows y and $y + 1$ and in the interior of the loop lies between edges $((x_i, y), (x_i, y + 1))$ and $((x_{i+1}, y), (x_{i+1}, y + 1))$ for some odd i, and, moreover, every vertex on row y of the grid, not in G, and in the interior of the loop lies between x_i and x_{i+1} for some odd i. Hence the number of squares between edges $((x_i, y), (x_i, y+1))$ and $((x_{i+1}, y), (x_{i+1}, y+1))$ for i odd equals the number of squares which are both between rows y and $y + 1$ and in the interior of the loop, and, furthermore, the number of vertices not in G lying on row y between x_i and x_{i+1} for i odd equals the number of vertices not in G lying on row y and in the interior of the loop. Thus

$$\prod_{j=1}^{m} a_{J_0(x_j,y_0),J_1(x_j,y_0+1)} = \prod_{j=1}^{m_0} \frac{a_{J_0(x_{2j},y_0),J_1(x_{2j},y_0+1)}}{a_{J_0(x_{2j-1},y_0),J_1(x_{2j-1},y_0+1)}} \tag{14}$$

equals (-1) to the power of the number of squares which are both in the interior of the loop and between rows y and $y+1$ minus the number of such vertices in the interior on row y. Hence, the product of this expression over all rows, that is,

$$\frac{\prod_{i \in I} a_{i,\pi(i)}}{\prod_{i \in I} a_{i,\pi\prime(i)}} \tag{15}$$

equals (-1) to the power of the number of squares in the interior of the loop minus the number of vertices lying on row y and in the interior, but not in G.

Therefore

$$(-1)^{\frac{\mathrm{len}(\ell)}{2}-1} \frac{\prod_{i \in I(\ell)} a_{i,\pi(i)}}{\prod_{i \in I(\ell)} a_{i,\pi\prime(i)}} \tag{16}$$

is equal to (-1) to the power of the length of the loop, divided by 2, minus 1, plus the number of squares inside the loop, minus the number of vertices in the grid and inside the loop, but not in G. Pick's theorem states that, for polygons whose vertices belong to a square grid, $A = I + B/2 - 1$, where A is the area enclosed by the polygon, I is the number of grid points inside the polygon, and B is the number of grid points on the boundary of the polygon. Hence our expression is equal to (-1) to the power of the number of vertices inside the loop, minus the number of vertices in the grid and inside the loop, but not in G. This is the same as the number of vertices of G inside the loop. Now, the vertices inside the loop are matched only among themselves, not with the vertices outside the loop, in either M or $M\prime$. For a set of vertices to be matched only among themselves, there must be an even number of vertices in the set. Therefore the number of vertices of G inside the loop must be even, and (-1) raised to the power of this number must be 1.

Hence

$$(-1)^{\frac{\mathrm{len}(\ell)}{2}-1} \frac{\prod_{i \in I(\ell)} a_{i,\pi(i)}}{\prod_{i \in I(\ell)} a_{i,\pi\prime(i)}} = 1 \tag{17}$$

as we desired to prove. We conclude, by taking the product over all loops $\ell \in L$, that

$$\frac{\mathrm{sgn}(\pi) \prod_{i=1}^{n} a_{i,\pi(i)}}{\mathrm{sgn}(\pi\prime) \prod_{i=1}^{n} a_{i,\pi\prime(i)}} = 1. \tag{18}$$

Therefore the terms of

$$\det(K) = \sum_{\pi \in P(\{1,2,\ldots,n\})} \mathrm{sgn}(\pi) \prod_{i=1}^{n} a_{i,\pi(i)} \tag{19}$$

all have the same sign. It follows that the number of perfect matchings, that is, of permutations π for which $\prod_{i=1}^{n} a_{i,\pi(i)}$ is non-zero, is equal to the absolute value of $\det(K)$.

2.3. **Why does K^{-1} give us the probabilities of patterns?** Let us have a graph G and a subgraph H whose every vertex has degree 1. There is a one-to-one correspondence between perfect matchings of G having H as a subgraph and perfect matchings of $G-H$. (By $G-H$ we mean the graph (V, E), where V is the set of all vertices of G not in H and E is the set of all edges of G between two vertices of G not in H.) Therefore the probability that a random perfect matching of G have the set of edges of H as a subset is equal to the number of perfect matchings of $G-H$ divided by the number of perfect matchings of G.

What is, then, the number of perfect matchings of $G-H$? One way to compute it is to construct a Kasteleyn-Wilson matrix for $G-H$. Another way, which will soon prove its virtues, is to take the minor of the Kasteleyn-Wilson matrix for G resulting from the deletion of the rows and columns corresponding to H. Certainly, this minor is not the same as the Kasteleyn-Wilson matrix for $G-H$; the signs of the matrix entries are different. Nevertheless, the absolute value of the determinant of the minor is equal to the number of perfect matchings of $G-H$. To prove this, we need to do the same as in the previous section, namely, show that all non-zero terms of the expression of the determinant as a sum over permutations have the same sign. If we proceed in the same way as before, we arrive at a point where the only difference is that we have (-1) to the power of the number of vertices of G inside a loop instead of the number of vertices of $G-H$ inside the same loop. This difference is no difference if there is an even number of vertices of H inside the loop. Since the vertices of H inside the loop cannot be connected with the vertices of H outside the loop, the vertices of H inside the loop are paired among themselves. Thus we have, as we wanted, that there is an even number of vertices of H inside the loop. This is enough for us to show that the terms in the expression of the minor of $K(G)$ as a sum over permutations do not cancel.

Therefore the number of perfect matchings of $G-H$ is equal to the absolute value of the determinant of the minor of $K(G)$ lacking the rows and columns corresponding to the vertices of H. If we assign the same probability to every perfect matching of G, the probability that a random perfect matching will have H as a subgraph will be equal to the number of perfect matchings of $G-H$ divided by the number of perfect matchings of G, that is, to the absolute value of the determinant of the aforementioned minor of $K(G)$ divided by the absolute value of the determinant of $K(G)$. By Jacobi's rule, a corollary of Cramer's rule, this is equal to the absolute value of the determinant of the minor of $(K(G)^{-1})^T$ consisting of those rows and columns omitted from the minor in the numerator, that is, of rows $1 \leq b_1 < b_2 < \cdots b_m \leq n$ and columns $1 \leq c_1 < c_2 < \cdots c_m \leq n$, where $v_{b_1}, v_{b_2}, \cdots v_{b_m}$ and $w_{c_1}, w_{c_2}, \cdots w_{c_m}$ are the vertices of H.

This is a clear improvement over the expression $\frac{\|K(G-H)\|}{\|K(G)\|}$. Instead of dealing with determinants of the size of G, we deal with a determinant of the size of H. As we explained in the introduction, we are interested in finding the probability of small subgraphs, or "local patterns", in a large graph G. We want to find what happens when we have an infinite sequence of G's whose number of vertices goes to infinity. Now it is enough for us to examine a fixed number of entries in each $(K(G_i)^{-1})^T$, and determine their asymptotic behavior as $i \to \infty$.

2.4. **How can we find the entries of K^{-1} for an Aztec diamond by counting perfect matchings?** Suppose that we have to compute the determinant of the minor of $K(G)^{-1}$ consisting of rows $1 \leq b_1 < b_2 < \cdots < b_m \leq n$ and columns $1 \leq c_1 < c_2 < \cdots <$

$c_m \leq n$. To do so, we have to compute the entry $((K(G)^{-1})^T)_{b_i,c_j}$ for $1 \leq i,j \leq m$. In other words, in order to compute the probability of any local pattern, it suffices to be able to compute an arbitrary entry of the inverse Kasteleyn matrix. If we have a sequence of graphs $\{G_k\}_{k=1}^{\infty}$ (such as, for example, Aztec diamonds of higher and higher order) we will know the asymptotic behavior of the probabilities of local patterns if we know the asymptotic behavior of the entries in the sequence of matrices $\{K(G_k)^{-1}\}_{k=1}^{\infty}$.

We can, of course, compute a first minor[1] of $K(G)$, and then apply Cramer's rule, whenever we want an entry of $(K(G)^{-1})^T$. Unfortunately, it seems very hard to obtain asymptotic expressions directly from the minors. We will reduce the problem of computing the minor of $K(G)$ resulting from the deletion of row i and column j to an enumerative problem whose solution we will be able to represent in a form other than a determinant. It would seem, at first sight, that we can use the same kind of argument we used to answer the previous question, and show that such a minor is equal (up to sign) to the number of all perfect matchings of G with white vertex i and black vertex j deleted. Unfortunately, this is not the case. A first minor without row i and column j is equal to a sum whose number of terms is equal to number of perfect matchings of G with white vertex i and black vertex j deleted. The problem is that, while every term has absolute value 1, not every term has the same sign. It is easy to show, by the same kind of reasoning we employed in our answer to the first question, that the matter of whether or not two terms have the same sign can be determined by examining the loops in the superimposition of the two matchings corresponding to the two terms. The terms have the same sign if and only if there is an even number of loops having one of the two deleted vertices, but not the other, in their interiors.

Definition 1. The *Aztec diamond* of order n is a planar graph consisting of vertices

$$\{(2r+1,2s) : 0 \leq r < n, 0 \leq s \leq n\} \cup \{(2r,2s+1) : 0 \leq r \leq n, 0 \leq s < n\} \tag{20}$$

and of edges

$$\begin{aligned}\{\{\{(2r,2s+1),(2r+1,2s+2)\},\{(2r+1,2s+2),(2r+2,2s+1)\},\\ \{(2r+2,2s+1),(2r+1,2s)\},\{(2r+1,2s),(2r,2s+1)\}\} : 0 \leq r < n, 0 \leq s < n\}\end{aligned} \tag{21}$$

in Cartesian coordinates.

It will soon become apparent that, for, our purposes, the system of coordinates in Figure 17 is more convenient than Cartesian coordinates. It will also become clear why we draw the Aztec diamond as if on an infinite square grid tilted 45 degrees from the "natural" direction. For now, let us notice that, if we color the Aztec diamond as a checkerboard, all vertices on a column have the same color, as do all vertices on a row. Let us also use the system of coordinates in Figure 17 instead of the Cartesian system, and refer to the edge consisting of white vertex (x_0, y_0) and black vertex (x_1, y_1) as $((x_0, y_0), (x_1, y_1))$.

We can reformulate the rule for comparing terms' signs so that it does not mention loops.

Lemma 1. *Let A and B be two perfect matchings of the Aztec diamond of order n with the white vertex at $(w_0, w_1 + d_1)$ and the black vertex at $(w_0 + d_0, w_1)$ deleted*[2]*. The following two conditions are equivalent:*

[1] That is, a minor that has all columns of the matrix of which it is a minor, but one, and all rows but one.

[2] We shall henceforth use the system of coordinates in Figure 17 instead of the Cartesian system.

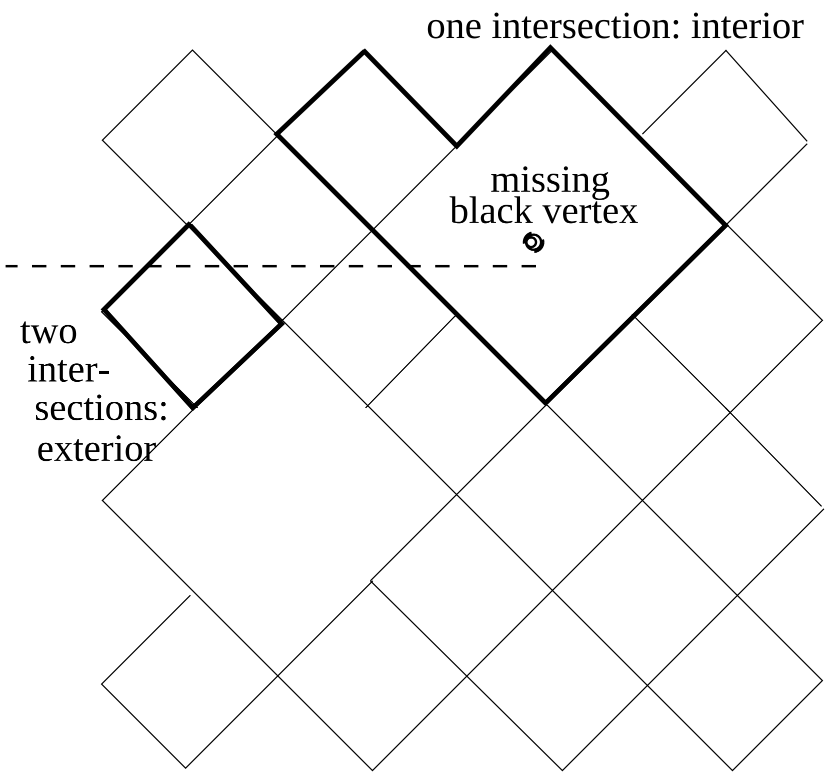

FIGURE 15. Counting intersections

1. $A \cup B - A \cap B$ *has an even number of loops containing exactly one of the two deleted vertices.*
2. $w(A) \cong w(B) \mod 2$*, where* $w(T)$ *is the number of edges of the form* $((i-1, w_1+1),(i,w_1))$*,* $1 < i < w_0+d_0$*;* $((i,w_1+1),(i,w_1))$*,* $1 \geq i < w_0+d_0$*;* $((i,w_1+d_1),(i,w_1+d_1-1))$*,* $1 \geq i < w_0$*, and* $((i,w_1+d_1),(i+1,w_1+d_1-1))$*,* $1 \geq i < w_0$ *in a perfect matching* T*.*

Proof. The sum $w(A)+w(B)$ is congruent, modulo 2, to the total number of edges in $A \cup B - A \cap B$ consisting of a black vertex $(x,y) \in D$ and white vertex $(x-1,y+1)$ or $(x,y+1)$. Given a loop ℓ in $A \cup B - A \cap B$, the number of edges in it consisting of a black vertex $(x,y) \in E$ and white vertex $(x-1,y+1)$ or $(x,y+1)$ is equal to the number of times the loop crosses a ray with its end slightly below the deleted black vertex and with diagonal direction with respect to the square grid. (See figure 15.) This number is even if the deleted black vertex is in the exterior of the loop, and odd if it is in the interior. The same holds for F and the deleted white vertex. Hence a loop has an even number of edges consisting of a black vertex $(x,y) \in D$ and white vertex $(x-1,y+1)$ or $(x,y+1)$ if and only if it has exactly one of the two deleted vertices in its interior. We conclude the proof by summing over all loops. ☐

It follows that $|\sum(-1)^{w(T)}| = |\sum_\pi \operatorname{sgn}(\pi)\prod_{i=1}^n a_{i,\pi(i)}|$, where the sum on the left is over all perfect matchings T of the Aztec diamond with white vertex at (w_0, w_1+d_1) and black vertex

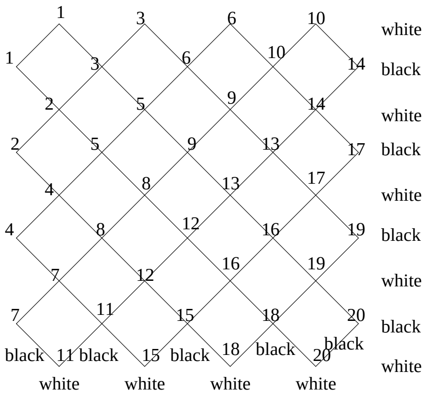

FIGURE 16. Ordering

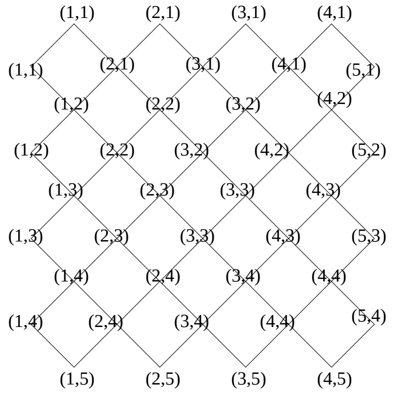

FIGURE 17. Coordinates

at (w_0+d_0, w_1), and $|a_{i,j}|_1^n$ is the first minor of the Kasteleyn matrix $K(G)$ of the (intact) Aztec diamond G where the row corresponding to the white vertex at (w_0, w_1+d_1) and the column corresponding to the black vertex at (w_0+d_0, w_1). Thus, if we compute $|\sum(-1)^{w(T)}|$ (and this is essentially an enumerative problem, which we shall solve enumeratively), we will know the absolute value of $\sum_\pi \operatorname{sgn}(\pi)\prod_{i=1}^n a_{i,\pi(i)}$, which will give us the absolute value of the entry of $(K(G)^{-1})^T$ to be computed, but not its sign. We will find the sign now.

It is better, for this particular goal, to express the entry of the inverse Kasteleyn matrix of the Aztec diamond, not as the ratio of the determinant of the minor of the Kasteleyn matrix resulting from the deletion of column i_0 and row j_0, divided by the determinant of the Kasteleyn matrix, multiplied by $(-1)^{i_0+j_0}$, but rather as the ratio of the determinant $|b_{i,j}|_1^{n-1}$ to the determinant of the Kasteleyn matrix $K(G) = |a_{i,j}|_1^n$, where $b_{i_0,j} = 0$ for $j \neq j_0$, $b_{i,j_0} = 0$ for $i \neq i_0$, $b_{i_0,j_0} = 1$, $b_{i,j} = a_{i,j}$ for $i \neq i_0$, $j \neq j_0$. We can then ask whether, for a permutation π of $\{1, 2, \cdots, n-1\}$, $\operatorname{sgn}(\pi)\cdot\prod_i b_{i,\pi(i)}$ has the same sign as $(-1)^{w(T)}$, where T is the perfect matching corresponding to π. (The answer will be the same for all permutations, so we have to ask it for only one permutation (or matching) we choose.) If the signs are the same, then $\sum(-1)^{w(T)} = (((K(G))^{-1})^T)_{i_0,j_0}\,|K(G)|_1^n$; if the sign are different, then $\sum(-1)^{w(T)} = -(((K(G))^{-1})^T)_{i_0,j_0}\,|K(G)|_1^n$.

For our search for the sign, we need to fix an ordering of the black and white vertices of the Aztec diamond. The ordering in Figure 16 will prove itself convenient.

We want to prove that, for any (w_0, w_1, d_0, d_1),

$$\sum(-1)^{w(T)} = (-1)^{d_0+d_1+1}\cdot\sum_\pi \operatorname{sgn}(\pi)\prod_{i=1}^n a_{i,\pi(i)}.$$

The most straightforward method of proof consists of comparing the signs of $\sum(-1)^{w(T)}$ and $\operatorname{sgn}(\pi)\prod_{i=1}^n a_{i,\pi(i)}$ for some tiling T and its corresponding permutation π. Unfortunately, this is also quite cumbersome, as which tilings are possible depends on the order of w_0, w_0+d_0, w_1 and w_1+d_1. In order not to bore the reader with twenty-four different cases, we will proceed explicitly only for the six cases corresponding to $d_0, d_1 > 0$, which can be treated as four different cases.

2.4.1. *Case 1:* $w_1 \leq w_0 < w_1 + d_1$. We choose to examine the matching having /-edges at $((i, w_1), (i, w_1))$, $w_1 \leq i < w_0 + d_0$, $((w_1, j), (w_1, j))$, $w_1 < j < w_1 + d_1$, and $((i, w_1 + d_1), (i + 1, w_1 + d_1 - 1))$, $w_1 \leq i < w_0$, where $((x_0, y_0), (x_1, y_1))$ denotes the edge consiting of a white vertex at (x_0, y_0) and a black vertex at (x_1, y_1). All other vertices are covered by \-edges: $((x, y), (x + 1, y))$ for $x \geq y$, given that at most one of the two conditions $w_1 \leq x < w_0 + d_0$, $y = w_1$ is fulfilled, and $((x, y), (x, y - 1))$ for $x < y$, given that at most one of $x = w_1$, $w_1 < y < w_1 + d_1$ holds, as well as at most one of $w_1 \leq x < w_0$, $y = w_1 + d_1$. (Note that we denote by /-edges and \-edges what we called "vertical edges" and "horizontal edges" before. This is just a change in notation with the purpose that the fact that we now draw the Aztec diamond tilted 45 degrees will not confuse the reader.) It is easy to verify that this is a perfect matching. For this matching, $(-1)^{w(T)}$ is $(-1)^{w_0+w_1}$.

Each matching corresponds to a permutation of $\{1, 2, \cdots, n\}$. (We have permutations of $\{1, 2, \cdots, n\}$, and not of $\{1, 2, \cdots, n-1\}$, because we are working with $|b_{i,j}|_1^n$, and not with a first minor of $K(G) = |a_{i,j}|_1^n$. The convenience of this choice, which we made without justification, will be clear by the end of this paragraph.) The permutation π corresponding to the matching we are now examining consists of one cycle. The length of this cycle is equal to the number of vertical lozenges, plus one. (This is not true in general. However, it is true in cases similar to the one we are currently examining, in which white vertices $\{f_1, f_2, \cdots f_m\}$ and black vertices $\{g_1, g_2, \cdots g_m\}$, which would be paired in the form (f_i, g_i), $1 \leq i \leq m$ in the all-\ matching, are paired in the form (f_i, g_{i+1}), $1 \leq i < m$, and vertices f_m and g_1 are missing.) Hence $\mathrm{sgn}(\pi)$ is equal to (-1) to the power of the number of vertical lozenges, that is, $2(w_0 - w_1) + d_0 + d_1 - 1$ lozenges.

We now need only examine $\prod_{i=1}^m b_{i,\pi(i)}$. Among the edges in our perfect matching, only $((i, w_1 + d_1), (i + 1, w_1 + d_1 - 1))$, $w_1 \leq i < w_0$, correspond to entries equal to (-1) in the determinant $|b_{i,j}|$. Therefore $\prod_{i=1}^m b_{i,\pi(i)} = (-1)^{w_0 - w_1}$.

We conclude that our matching T, for which $(-1)^{w(T)} = (-1)^{w_0+w_1-1}$, contributes $(-1)^{2(w_0-w_1)+d_0+d_1-1} \cdot (-1)^{w_0-w_1}$, to the determinant. Therefore an arbitrary matching T contributes

$$(-1)^{w(T)-(w_0+w_1-1)} \cdot (-1)^{2(w_0-w_1)+d_0+d_1-1} \cdot (-1)^{w_0-w_1}, \tag{22}$$

that is,

$$(-1)^{w(T)}(-1)^{d_0+d_1+1}, \tag{23}$$

to the determinant. Therefore

$$\sum (-1)^{w(T)} = (-1)^{d_0+d_1+1} \cdot \sum_{\pi} \mathrm{sgn}(\pi) \prod_{i=1}^{n} a_{i,\pi(i)}$$

for case 1.

2.4.2. *Case 2:* $w_0 \geq w_1 + d_1$. We choose to examine the matching having //-edges at $((i, w_1), (i, w_1))$, $w_1 < i < w_0 + d_0$, at $((w_0 + 1, j), (w_0, j + 1))$, $w_1 + d_1 \leq j \leq w_0$, at $((w_1, j), (w_1, j))$, $w_1 \leq j \leq w_0$, and at $((i + 1, w_0), (i, w_0 + 1))$, $w_1 \leq i < w_0$. All other vertices are covered by \-edges. The $(-1)^{w(T)}$ of this matching is $(-1)^{d_1+1}$. The sign of the permutation is

$$(-1)^{(w_0+d_0-w_1-1)+(w_0-(w_1+d_1)+1)+(w_0-w_1+1)+(w_0-w_1)}, \tag{24}$$

that is, $(-1)^{d_0+d_1+1}$. The product $\prod_{i=1}^{m} b_{i,\pi(i)}$ is $(-1)^{(w_0-(w_1+d_1)+1)+(w_0-w_1)}$, that is, $(-1)^{d_0+d_1}$. Hence each matching T contributes

$$(-1)^{w(T)}(-1)^{d_0+d_1+1} \tag{25}$$

to the determinant. Therefore

$$\sum(-1)^{w(T)} = (-1)^{d_0+d_1+1} \cdot \sum_{\pi} \operatorname{sgn}(\pi) \prod_{i=1}^{n} a_{i,\pi(i)}$$

for case 2.

2.4.3. *Case 3:* $w_0 < w_1 < w_0 + d_0$. We choose to examine the matching having //-edges at $((w_0, j), (w_0, j))$, $w_0 \le j < w_1 + d_1$, at $((i, w_0), (i, w_0))$, $w_0 < i \le w_0 + d_0$, and at $((w_0 + d_0, j + 1), (w_0 + d_0 + 1, j))$, $w_0 \le j < w_1$. All other vertices are covered by \-edges. The $(-1)^{w(T)}$ of this matching is $(-1)^{w_0+w_1+1}$. The sign of the permutation is $(-1)^{d_0+d_1}$. The product $\prod_{i=1}^{m} b_{i,\pi(i)}$ is $(-1)^{w_1-w_0}$. Therefore each matching T contributes

$$(-1)^{w(T)} \cdot (-1)^{d_0+d_1+1} \tag{26}$$

to the determinant. Therefore

$$\sum(-1)^{w(T)} = (-1)^{d_0+d_1+1} \cdot \sum_{\pi} \operatorname{sgn}(\pi) \prod_{i=1}^{n} a_{i,\pi(i)}$$

for case 3.

2.4.4. *Case 4:* $w_1 \ge w_0 + d_0$. We choose to examine the matching having //-edges at $((w_0, j), (w_0, j))$, $w_0 \le j < w_1 + d_1$, at $((i, w_0), (i, w_0))$, $w_0 < i \le w_1$, at $((w_1, j + 1), (w_1 + 1, j))$, $w_0 \le j \le w_1$, and at $((i, w_1 + 1), (i + 1, w_1))$, $w_0 + d_0 \le i < w_1$. All other vertices are covered by \-edges. The $(-1)^{w(T)}$ of this matching is $(-1)^{d_0+1}$. The sign of the permutation is $(-1)^{d_0+d_1+1}$. The product $\prod_{i=1}^{m} b_{i,\pi(i)}$ is $(-1)^{d_0+1}$. Therefore each matching T contributes

$$(-1)^{w(T)}(-1)^{d_0+d_1+1} \tag{27}$$

to the determinant. Therefore

$$\sum(-1)^{w(T)} = (-1)^{d_0+d_1+1} \cdot \sum_{\pi} \operatorname{sgn}(\pi) \prod_{i=1}^{n} a_{i,\pi(i)}$$

for case 4.

3. An enumerative problem

Consider an Aztec diamond of side, or order, n, with black vertex $(w_0 + d_0, w_1)$ and white vertex $(w_0, w_1 + d_1)$ missing. Our task in this section is to compute

$$\sum_{T}(-1)^{w(T)} \tag{28}$$

where T ranges over all perfect matchings of this Aztec diamond with two missing vertices. Enumeratively speaking, we must count the number of perfect matchings, counting as "negative" any matching T for which $(-1)^{w(T)} = (-1)$. The function $w(T)$, as defined in the previous section, gives, for a perfect matching T, the number of edges consisting of a black vertex $(x, y) \in D$ and white vertex $(x-1, y+1)$ or $(x, y+1)$. D is the union of two set of black vertices: $E = \{(i, w_1) : i < w_0 + d_0\}$ and $F = \{(i, w_1 + d_1) : i \leq w_0\}$.

Throughout this section, we will assume $d_0 > 0$, $d_1 > 0$. At the end it will become clear that this assumption involves no loss of generality. One of our tools for attacking the special case $d_0 > 0$, $d_1 > 0$ will be the EKLP Lemma, a classical result in [3] which we will soon explain. First of all, we must define two kinds of subsets of the Aztec diamond. They will be our intermediate objects of study.

Definition 2. An $n \times m$ *black-edged Aztec rectangle* with dents at $1 \leq x_1 < x_2 < ... < x_m \leq n+1$ is a graph consisting of white vertices (i, j), $1 \leq i \leq n$, $1 \leq j \leq m$, black vertices (i, j), $1 \leq i \leq n+1$, $1 \leq j \leq m-1$, and black vertices (i, m) such that $\forall 1 \leq k \leq m$ we have $i \neq x_k$. An $n \times m$ *white-edged Aztec rectangle* with teeth at $1 \leq y_1 < y_2 < ... < y_m \leq n$ is a graph consisting of white vertices (i, j), $1 \leq i \leq n$, $1 \leq j \leq m$, black vertices (i, j), $1 \leq i \leq n+1$, $1 \leq j \leq m$, and white vertices $(i, m+1)$ s.t. $\exists 1 \leq k \leq m$ with $i = x_k$.

Let the number of perfect matchings of an $n \times m$ black-edged Aztec rectangle with dents at $1 \leq x_1 < x_2 < ... < x_m \leq n+1$ be called $D_{n,m}(x_1, x_2, ..., x_m)$. What is the number of perfect matchings of an $n \times m$ white-edged Aztec rectangle Y with teeth at $1 \leq y_1 < y_2 < ... < y_m \leq n$? Every white vertex (or "tooth") $(y_i, m+1)$ must be covered by a horizontal or vertical edge, which will also cover black vertex (y, m) or $(y+1, m)$, respectively. Thus, any matching of Y is composed of

- a matching of a black-edged $n \times m$ Aztec rectangle with dents at $1 \leq x_1 < x_2 < ... < x_m \leq n+1$, where $x_i - y_i$ is either 0 or 1;
- the unique matching of the remaining region.

Therefore the number of matchings of Y is the m-fold sum

$$E_{n,m}(y_1, ..., y_m) = \sum_{x_i = y_i \text{ or } y_i + 1} D_{n,m}(x_1, ..., x_m) \tag{29}$$

where we assume that $D_{n,m}(x_1, x_2, ..., x_m) = 0$ when any two x_i's are equal.

In the same way that, given $D_{n,m}$, we have found $E_{n,m}$, given $E_{n,m}$, we can find $D_{n,m+1}$. We have again an m-fold sum:

$$D_{n,m+1}(x_1, ..., x_{m+1}) = \sum_{x_i \leq y_i < x_{i+1}} E_{n,m}(y_1, ..., y_m) \tag{30}$$

where we assume that $E_{n,m}(x_1, x_2, ..., x_m) = 0$ when any two x_i's are equal.

We will now be able to prove the following by induction. The proof in [3] does not use Lemma 3 explicitly.

Lemma 2 (EKLP Lemma)**.** *The number of matchings of an $n \times m$ black-edged Aztec rectangle with dents $1 \leq x_1 < x_2 < \cdots < x_m \leq n+1$ is*

$$D_{n,m}(x_1, x_2, ..., x_m) = \frac{2^{\frac{m(m-1)}{2}}}{(m-1)!!} \left| x_i^{j-1} \right|_1^m, \tag{31}$$

where $k!! = 1!\,2!\,3! \cdots (k-1)!\,k!$. *The number of matchings of an* $n \times m$ *white-edged Aztec rectangle with dents at* $1 \leq y_1 < y_2 < \cdots < y_m \leq n$ *is*

$$E_{n,m}(y_1, y_2, ..., y_m) = \frac{2^{\frac{m(m+1)}{2}}}{(m-1)!!} \left| y_i^{j-1} \right|_1^m \tag{32}$$

For the proof of this lemma, we need a special case of another lemma that we will later state in its full generality.

Lemma 3 (Lemma A (special case)). *Let* A *be an operator carrying polynomials to polynomials of the same or lesser degree:*

$$\begin{array}{lllll} x^0 & \rightarrow & & & a_{0,0}x^0 \\ x^1 & \rightarrow & & a_{1,1}x^1 + & a_{1,0}x^0 \\ x^2 & \rightarrow & a_{2,2}x^2 + & a_{2,1}x^1 + & a_{2,0}x^0 \\ \cdots & & & & \end{array} \tag{33}$$

Then

$$\left| A(x^{j-1})(x_i) \right|_1^m = a_{0,0} \cdot a_{1,1} \cdots a_{m-1,m-1} \cdot \left| x_i^{j-1} \right|_1^m, \tag{34}$$

where

$$A(x^j)(x_i) = a_{j,j}x_i^j + a_{j,j-1}x_i^{j-1} + \cdots + a_{j,0}x_i^0. \tag{35}$$

Proof. The determinant on the left side of (34) can be obtained from the determinant on the right side by elementary column operations. □

Proof of EKLP Lemma. The base case ($m = 1$) is trivial. The inductive step $D_{n,m} \rightarrow E_{n,m}$ follows directly from (30) and from Lemma A (special case).

The inductive step $E_{n,m} \rightarrow D_{n,m+1}$ requires some more work.

$$\begin{aligned} D_{n,m+1}(x_1, \cdots x_{m+1}) &= \sum_{x_i \leq y_i < x_{i+1}} E_{n,m}(y_1, ..., y_m) \\ &= \sum_{x_i \leq y_i < x_{i+1}} \frac{2^{\frac{m(m+1)}{2}}}{(m-1)!!} \left| y_i^{j-1} \right|_1^m \\ &= \frac{2^{\frac{m(m+1)}{2}}}{(m-1)!!} \left| x_i^{j-1} + (x_i+1)^{j-1} + \cdots (x_{i+1}-1)^{j-1} \right|_1^m \\ &= \frac{2^{\frac{m(m+1)}{2}}}{(m-1)!!} \left| \frac{1}{j}(B_j(x_{i+1}) - B_j(x_i)) \right|_1^m, \end{aligned} \tag{36}$$

where $B_r(y)$ is the Bernoulli polynomial of degree r, which has the property $\sum_{n=M}^{N-1} n^q = (B_{q+1}(N) - B_{q+1}(M))/(q+1)$. For a definition of $B_r(y)$, consult [11]. The only property of $B_r(y)$ we need to know is that it is a polynomial of degree r whose leading coefficient is one. We will adopt, for the sake of convenience, the convention $B_0(y) = y^0$. We can now continue:

$$D_{n,m+1}(x_1,\cdots x_{m+1}) = \frac{2^{\frac{m(m+1)}{2}}}{m!!} \left|B_j(x_{i+1}) - B_j(x_i)\right|_1^m \tag{37}$$

$$= \frac{2^{\frac{m(m+1)}{2}}}{m!!} \begin{vmatrix} 1 & B_1(x_1) & B_2(x^1) & \cdots & B_m(x_1) \\ 1 & B_1(x_2) & B_2(x_2) & \cdots & B_m(x_2) \\ \cdots \\ 1 & B_1(x_{m+1}) & B_2(x_{m+1}) & \cdots & B_m(x_{m+1}) \end{vmatrix}, \tag{38}$$

where we have pasted a new column onto the left edge of the determinant and a new row onto the top edge, and then added the first row to the second one, the second to the third one, and so on, in succession

$$= \frac{2^{\frac{m(m+1)}{2}}}{m!!} \left|x_i^{j-1}\right|_1^{m+1}, \tag{39}$$

by Lemma A (special case). □

Let us now count matchings with weights 1 and (-1): a matching may count as one matching or as minus one matchings. Consider an $n \times m$ black-edged Aztec rectangle with dents at $1 \le x_1 < x_2 < \cdots x_{m-1} \le n+1$, and, in addition, a dent at w_0. Let us multiply the total number of matchings by (-1) if there is an odd number of x_i's smaller than w_0. (Here we are counting either all matchings as positive matchings or all as negative matchings.) What is, then, this total, weighted number of matchings? Conveniently, it is

$$\frac{2^{\frac{m(m-1)}{2}}}{(m-1)!!} \begin{vmatrix} w_0^{j-1} \\ x_1^{j-1} \\ x_2^{j-1} \\ \cdots \\ x_{m-1}^{j-1} \end{vmatrix} \tag{40}$$

We have just taken (31) and shifted the w_0^j row corresponding to the dent at w_0 as many positions upwards as there are x_i's smaller than w_0.

What is the weighted number of matchings of a white-edged $n \times m$ Aztec rectangle Y with teeth at $y_1, y_2, \cdots y_{m-1}$ and a black hole at (w_0, m)? (The weight of each matching is (-1) to the power of the number of its edges consist of a black vertex (i, w_1) and white vertex $(i-1, w_1+1)$ or (i, w_1+1), where $1 \le i < w_0 + d_0$.) A cursory examination makes clear that a matching of Y will have weight 1 if and only if the black-edged $n \times m$ Aztec rectangle the matching outlines has an even number of dents with indices lower than w. Thus the weighted number of matchings is

$$\sum_{x_i = y_i or y_i+1} \frac{2^{\frac{m(m-1)}{2}}}{(m-1)!!} \begin{vmatrix} w_0^{j-1} \\ x_1^{j-1} \\ x_2^{j-1} \\ \cdots \\ x_{m-1}^{j-1} \end{vmatrix}, \tag{41}$$

which is the same as

$$\frac{2^{\frac{m(m-1)}{2}}}{(m-1)!!} \begin{vmatrix} w_0^{j-1} \\ y_1^{j-1} + (y_1+1)^{j-1} \\ y_2^{j-1} + (y_2+1)^{j-1} \\ \cdots \\ y_{m-1}^{j-1} + (y_{m-1}+1)^{j-1} \end{vmatrix}. \tag{42}$$

Here we face a difficulty. We would like to use Lemma 3. Unfortunately, we have w_0^{j-1} on the top row. What we need is to find something which is transformed into w_0^{j-1} by the linear operations taking a row of the form x^{j-1} to $x^{j-1} + (x+1)^{j-1}$. First of all, we need a stronger version of Lemma 3.

Lemma 4 (Lemma A, stronger version)**.** *Let A be an operator carrying polynomials to polynomials of the same or lesser degree:*

$$\begin{array}{llccc} x^0 & \to & & & a_{0,0}x^0 \\ x^1 & \to & & a_{1,1}x^1 + & a_{1,0}x^0 \\ x^2 & \to & a_{2,2}x^2 + & a_{2,1}x^1 + & a_{2,0}x^0 \\ \cdots & & & & \end{array} \tag{43}$$

Then

$$\left| \sum_{1 \le k \le l_i} b_{k,i} A(x^{j-1})(x_{k,i}) \right|_1^m = a_{0,0} \ldots a_{m-1,m-1} \left| \sum_{1 \le k \le l_i} b_{k,i} x_{k,i}^{j-1} \right|_1^m, \tag{44}$$

for any l_i, $b_{k,i}$, $1 \le k \le l_i$, $1 \le i \le m$, where

$$A(x^j)(x_i) = a_{j,j} x_i^j + a_{j,j-1} x_i^{j-1} + \cdots + a_{j,0} x_i^0. \tag{45}$$

Proof. The special case followed from the fact that the same sequence of column operations transforms the row $A(x^{j-1})(y)$ into y^j and the row $A(x^j)(z)$ into z^j, for any values y,z. Therefore the same sequence transforms the row $a \cdot A(x^j)(y) + b \cdot A(x^j)(z)$ into $a \cdot y^j + b \cdot z^j$. $\square$

We need a_k's such that $\sum_k a_k A(x^j)(v_k) = w_0^j$ for $0 \le j < m$, where A is the operator taking x^j to $x^j + (x+1)^j$.

Definition 3. Δ is the operator taking x^j to $(x+1)^j - x^j$.

Then $A = (2I + \Delta)$, where I is the identity operator, and we have, formally,

$$
\begin{aligned}
(2I+\Delta)^{-1} &= \frac{1}{2}\cdot(I+\frac{\Delta}{2})^{-1} \\
&= \frac{1}{2}\cdot\sum_{j=0}^{\infty}(-1)^j\frac{\Delta}{2}^j
\end{aligned}
\tag{46}
$$

Since Δ^j vanishes on polynomials of degree smaller than j, and we are dealing with polynomials of degree at most $m-1$, we can use just the first m terms of the series:

$$
(1/2)\sum_{j=0}^{m-1}(-1)^j(\Delta/2)^j \tag{47}
$$

The reader can easily check that, for any x^k with $k < m$,

$$
(2I+\Delta)((\frac{1}{2}\sum_{j=0}^{m-1}(-1)^j(\frac{\Delta}{2})^j)(x^k)) \tag{48}
$$

gives x^k plus a constant times $\Delta^m x^k$, and, since $m > k$, $\Delta^m x^k$ is zero. One can also check the same for

$$
((\frac{1}{2}\sum_{j=0}^{m-1}(-1)^j(\frac{\Delta}{2})^j)(x^k))(2I+\Delta). \tag{49}
$$

It is quite convenient that $(2I + \Delta)$ and

$$
((\frac{1}{2}\sum_{j=0}^{m-1}(-1)^j(\frac{\Delta}{2})^j)(x^k)) \tag{50}
$$

commute, and that their composition is the same as the identity operator for the domain we are interested in. We will use the shorthand $(2I + \Delta)^{-1}$ for

$$
((\frac{1}{2}\sum_{j=0}^{m-1}(-1)^j(\Delta/2)^j)(x^k)) \tag{51}
$$

without any compunctions.

If we define $a_0, a_1, \cdots, a_{m-1}$ by

$$
(\frac{1}{2}\sum_{j=0}^{m-1}(-1)^j(\Delta/2)^j)(f(x)) = \sum_{i=0}^{m-1}a_i f(x+i), \tag{52}
$$

then

$$\begin{aligned}\sum_{i=0}^{m-1} a_i((2I+\Delta)(x^j)(w_0+i)) &= (((\frac{1}{2}\sum_{j=0}^{m-1}(-1)^j(\Delta/2)^j)(2I+\Delta))(x^j))(w_0) \\ &= (x^j)(w_0) \\ &= w_0^j,\end{aligned} \tag{53}$$

for $0 \leq j < m$, as we desired.

Hence, by Lemma A,

$$\begin{vmatrix} w_0^{j-1} \\ y_1^{j-1} + (y_1+1)^{j-1} \\ y_2^{j-1} + (y_2+1)^{j-1} \\ \cdots \\ y_{m-1}^{j-1} + (y_{m-1}+1)^{j-1} \end{vmatrix} \tag{54}$$

is equal to

$$2^m \cdot \begin{vmatrix} (\frac{1}{2}\sum_{k=0}^{m-1}(-1)^k(\frac{\Delta}{2})^k)(x^{j-1})(w_0) \\ y_1^{j-1} \\ y_2^{j-1} \\ \cdots \\ y_{m-1}^{j-1} \end{vmatrix}, \tag{55}$$

or, in shorthand,

$$2^m \cdot \begin{vmatrix} ((2I+\Delta)^{-1}(x^{j-1}))(w_0) \\ y_1^{j-1} \\ y_2^{j-1} \\ \cdots \\ y_{m-1}^{j-1} \end{vmatrix} \tag{56}$$

Now, for every polynomial p,

$$p(w_0) = (((I+\Delta)^{w_0-1})(p))(1). \tag{57}$$

Hence we can write

$$2^m \cdot \begin{vmatrix} ((I+\Delta)^{w_0-1}((2I+\Delta)^{-1}(x^{j-1})))(1) \\ y_1^{j-1} \\ y_2^{j-1} \\ \cdots \\ y_{m-1}^{j-1} \end{vmatrix} \tag{58}$$

and eliminate all terms of degree m or higher. Hence the weighted number of matchings of a white-edged $n \times m$ Aztec rectangle with teeth at $y_1, y_2, \cdots y_{m-1}$ and a black hole at (w_0, m) is

$$\frac{2^{\frac{m(m+1)}{2}}}{(m-1)!!}\begin{vmatrix} ((I+\Delta)^{w_0-1}((2I+\Delta)^{-1}(x^{j-1})))(1) \\ y_1^{j-1} \\ y_2^{j-1} \\ \cdots \\ y_{m-1}^{j-1} \end{vmatrix} \tag{59}$$

What would be the weighted number of matchings of a black-edged $n \times (m+1)$ Aztec rectangle with dents at $x_1, x_2, \cdots x_m$ and a black hole at (w_0, m)? This is the sum we have to simplify:

$$\sum_{x_i \le y_i < x_{i+1}} \frac{2^{\frac{m(m+1)}{2}}}{(m-1)!!}\begin{vmatrix} ((I+\Delta)^{w_0-1}((2I+\Delta)^{-1}(x^{j-1})))(1) \\ y_1^{j-1} \\ y_2^{j-1} \\ \cdots \\ y_{m-1}^{j-1} \end{vmatrix} \tag{60}$$

This is equal to $\frac{2^{\frac{m(m+1)}{2}}}{(m-1)!!}$ times

$$\sum_{x_i \le y_i < x_{i+1}} \begin{vmatrix} ((I+\Delta)^{w_0-1}((2I+\Delta)^{-1}(x^{j-1})))(1) \\ y_1^{j-1} \\ y_2^{j-1} \\ \cdots \\ y_{m-1}^{j-1} \end{vmatrix} \tag{61}$$

$$= \begin{vmatrix} ((I+\Delta)^{w_0-1}((2I+\Delta)^{-1}(x^{j-1})))(1) \\ x_1^{j-1} + (x_1+1)^{j-1} + \cdots + (x_2-1)^{j-1} \\ x_2^{j-1} + (x_2+1)^{j-1} + \cdots + (x_3-1)^{j-1} \\ \cdots \\ x_{m-1}^{j-1} + (x_{m-1}+1)^{j-1} + \cdots + (x_m-1)^{j-1} \end{vmatrix} \tag{62}$$

$$= \begin{vmatrix} ((I+\Delta)^{w_0-1}((2I+\Delta)^{-1}(x^{j-1})))(1) \\ \frac{1}{j}(B_j(x_2) - B_j(x_1)) \\ \frac{1}{j}(B_j(x_3) - B_j(x_2)) \\ \cdots \\ \frac{1}{j}(B_j(x_m) - B_j(x_{m-1})) \end{vmatrix} \tag{63}$$

$$= \begin{vmatrix} 1 & \frac{1}{j-1}B_{j-1}(x_1) \\ 0 & ((I+\Delta)^{w_0-1}((2I+\Delta)^{-1}(x^{j-2})))(1) \\ 0 & \frac{1}{j-1}B_{j-1}(x_2) - B_{j-1}(x_1) \\ \cdots & \cdots \\ 0 & \frac{1}{j-1}B_{j-1}(x_m) - B_{j-1}(x_{m-1}) \end{vmatrix} \tag{64}$$

$$= - \begin{vmatrix} 0 & ((I+\Delta)^{w_0-1}((2I+\Delta)^{-1}(x^{j-2})))(1) \\ 1 & \frac{1}{j-1}B_{j-1}(x_1) \\ 0 & \frac{1}{j-1}B_{j-1}(x_2) - B_{j-1}(x_1) \\ \cdots & \cdots \\ 0 & \frac{1}{j-1}B_{j-1}(x_m) - B_{j-1}(x_{m-1}) \end{vmatrix} \tag{65}$$

$$= - \begin{vmatrix} 0 & ((I+\Delta)^{w_0-1}((2I+\Delta)^{-1}(x^{j-2})))(1) \\ 1 & \frac{1}{j-1}B_{j-1}(x_1) \\ 1 & \frac{1}{j-1}B_{j-1}(x_2) \\ \cdots & \cdots \\ 1 & \frac{1}{j-1}B_{j-1}(x_m) \end{vmatrix} \tag{66}$$

Since $\{((I+\Delta)^{w_0-1}((2I+\Delta)^{-1}(x^{j-2})))(1)\}_{j=2,3,\cdots m+1}$ is a linear combination of rows of the form $\{k^{j-2}\}_{j=2,3,\cdots m+1}$, $1 \le k \le m$, it is enough to show how to simplify

$$\begin{vmatrix} 0 & k^{j-2} \\ 1 & \frac{1}{j-1}B_{j-1}(x_1) \\ 1 & \frac{1}{j-1}B_{j-1}(x_2) \\ \cdots & \cdots \\ 1 & \frac{1}{j-1}B_{j-1}(x_m) \end{vmatrix} \tag{67}$$

This happens to be easier than one would expect. As a particular case of

$$\sum_{n=M}^{N-1} n^q = \frac{B_{q+1}(N) - B_{q+1}(M)}{q+1},$$

we have

$$k^{j-1} = \frac{1}{j}(B_j(k+1) - B_j(k)).$$

Therefore

$$\begin{vmatrix} 0 & k^{j-2} \\ 1 & \frac{1}{j-1}B_{j-1}(x_1) \\ 1 & \frac{1}{j-1}B_{j-1}(x_2) \\ \cdots & \cdots \\ 1 & \frac{1}{j-1}B_{j-1}(x_m) \end{vmatrix} = \begin{vmatrix} 1-1 & \frac{1}{j-1}(B_{j-1}(k+1) - B_{j-1}(k)) \\ 1 & \frac{1}{j-1}B_{j-1}(x_1) \\ 1 & \frac{1}{j-1}B_{j-1}(x_2) \\ \cdots & \cdots \\ 1 & \frac{1}{j-1}B_{j-1}(x_m) \end{vmatrix} \tag{68}$$

$$= \frac{1}{m!}\begin{vmatrix} (k+1)^{j-1} - k^{j-1} \\ x_1^{j-1} \\ x_2^{j-1} \\ \cdots \\ x_m^{j-1} \end{vmatrix} \tag{69}$$

$$= \frac{1}{m!}\begin{vmatrix} \Delta(x^{j-1})(k) \\ x_1^{j-1} \\ x_2^{j-1} \\ \cdots \\ x_m^{j-1} \end{vmatrix} \tag{70}$$

Thus, we now know that the sequence of elementary column operations we have to apply to the matrix having x_i^j on its lower rows in order to make it into a matrix having $\{1, \frac{1}{1}B_1(x_i), \frac{1}{2}B_2(x_i), \cdots \frac{1}{m}B_m(x_i)\}$ on its lower rows transforms the row $\{\Delta(x^{j-1})(k)\}_1^{m+1}$ into the row $\{0, k^0, k^1, \cdots k^m\}$. What row would be transformed into the row

$$\{0,((I+\Delta)^{w_0-1}((2I+\Delta)^{-1}(x^0))(1), ((I+\Delta)^{w_0-1}((2I+\Delta)^{-1}(x^1))(1), \cdots, \\ ((I+\Delta)^{w_0-1}((2I+\Delta)^{-1}(x^{m-1}))(1)\}?$$

Let us express $(I+\Delta)^{w_0-1}((2I+\Delta)^{-1})(f)(x)$ (f any polynomial of degree at most $m-1$) in the form $\sum_{k=0}^{m-1} a_k f(x+k)$ for some numbers $a_1, a_2, \cdots a_m$. If rows $y_1, y_2, \cdots y_r$ are transformed into rows $z_1, z_2, \cdots z_r$, respectively, then the row $\sum_{i=1}^r b_i y_i$ must be transformed into the row $\sum_{i=1}^r b_i z_i$. Therefore the row $\{\sum_{k=0}^{m-1} a_k(\Delta(x^{j-1}))(k+1)\}_1^{m+1}$ is transformed into

$$\{0, \sum_{k=0}^{m-1} a_k(k+1)^0, \sum_{k=0}^{m-1} a_k(k+1)^1, \cdots, \sum_{k=0}^{m-1} a_k(k+1)^{m-1}\},$$

which is the same as

$$\{0,(I+\Delta)^{w_0-1}((2I+\Delta)^{-1}(x^0))(1), (I+\Delta)^{w_0-1}((2I+\Delta)^{-1}(x^1))(1), \cdots, \\ (I+\Delta)^{w_0-1}((2I+\Delta)^{-1}(x^{m-1}))(1)\}.$$

Now, what is $\sum_{k=0}^{m-1} a_k(\Delta(x^j))(k+1)$ for $0 \leq j < m$? It is

$$((I+\Delta)^{w_0-1}((2I+\Delta)^{-1}(\Delta(x^j))))(1).$$

Hence the weighted number of matchings of a black-edged $n \times (m+1)$ Aztec rectangle with dents at $x_1, \cdots, x_m$ and a black hole at (w_0, m) is

$$-\frac{2^{\frac{m(m+1)}{2}}}{m!!}\begin{vmatrix} ((I+\Delta)^{w_0-1}(2I+\Delta)^{-1}\Delta)(x^{j-1})(1) \\ x_1^{j-1} \\ x_2^{j-1} \\ \cdots \\ x_m^{j-1} \end{vmatrix}. \tag{71}$$

In the same way we have arrived at this result, and using it as a base case for induction, it is easy to prove the following.

Lemma 5. *The weighted number of matchings of a white-edged $n \times (m+d_1)$ Aztec rectangle with teeth at $y_1, \cdots y_{m+d_1-1}$ and a black hole at (w_0, m) is*

$$(-1)^{d_1}\frac{2^{\frac{(m+d_1)(m+d_1+1)}{2}}}{(m+d_1-1)!!}\begin{vmatrix} ((I+\Delta)^{w_0-1}(2I+\Delta)^{-(d_1+1)}\Delta^{d_1})(x^{j-1})(1) \\ y_1^{j-1} \\ y_2^{j-1} \\ \cdots \\ y_{m+d_1-1}^{j-1} \end{vmatrix} \tag{72}$$

and that the weighted number of matchings of a black-edged $n \times (m+d_1)$ Aztec rectangle with dents at $x_1, x_2, \cdots, x_{m+d_1-1}$ and a black hole at (w_0, m) is

$$(-1)^{d_1}\frac{2^{\frac{(m+d_1)(m+d_1-1)}{2}}}{(m+d_1-1)!!}\begin{vmatrix} ((I+\Delta)^{w_0-1}(2I+\Delta)^{-d_1}\Delta^{d_1})(x^{j-1})(1) \\ y_1^{j-1} \\ y_2^{j-1} \\ \cdots \\ y_{m+d_1-1}^{j-1} \end{vmatrix}. \tag{73}$$

Notice that we have used the fact that $(2I+\Delta)^{-1}$ and Δ commute. They do so because $(2I+\Delta)^{-1}$, on the domain of polynomials of degree lower than a given bound, is shorthand for a finite sum of powers of Δ.

We can now attack our main objective, namely, the enumeration of matchings of an Aztec diamond with a black hole at (w_0+d_0, w_1) and a white hole at (w_0, w_1+d_1), where some matchings are counted as negative matchings. Let us first consider the special case $w_0 = 1$. What happens at white and black vertices (x, y) with $x = 1$? The hole at $(1, w_1+d_1)$ forces a zig-zag pattern covering all those vertices. (See figure 18.) None of the edges covering these vertices counts towards the weight of the matching. Thus, the weighted number of matchings of an Aztec diamond of order n with a black hole at $(1+d_0, w_1)$ and a white hole at $(1, w_1+d_1)$ is the same as the weighted number of matchings of a $(n-1)\times n$ white-edged Aztec rectangle with teeth at $\{1, 2, \cdots n\}$ and a black hole at (d_0, w_1). By Lemma 5, this is

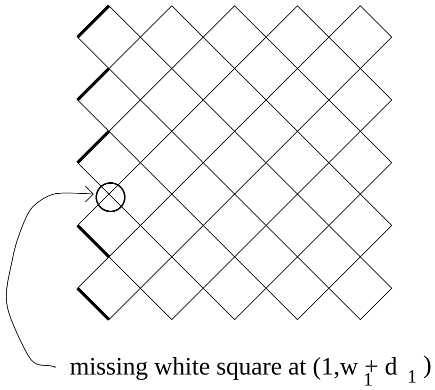

FIGURE 18. Forced zig-zag pattern

equal to

$$(-1)^{n-w_1}\frac{2^{\frac{n(n+1)}{2}}}{(n-1)!!}\begin{vmatrix} ((I+\Delta)^{d_0-1}(2I+\Delta)^{-(n-w_1+1)}\Delta^{n-w_1})(x^{j-1})(1) \\ 1^{j-1} \\ 2^{j-1} \\ \cdots \\ (n-1)^{j-1}\end{vmatrix}. \tag{74}$$

The top row is a linear combination of rows of the form $\{\Delta^k(x^{j-1})(1)\}$, $1 \leq j \leq n$. All such rows with $k > (n-1)$ vanish. All such rows with $k < (n-1)$ are linear combinations of the rows

$$\{i^{j-1}\}, 1 \leq j \leq n, 1 \leq i \leq n-1,$$

Since an n by n determinant whose last $n-1$ rows are $\{1^j\}, \{2^j\}, \cdots \{(n-1)^j\}$ and whose first row is one of 3 vanishes, we can discard all terms of the form

$$\{\Delta^k(x^{j-1})(1)\}, 1 \leq j \leq n, k < (n-1)$$

from the top row of the determinant in (74). Hence we must consider only the term of the form

$$\{C \cdot \Delta^{n-1}(x^{j-1})(1)\}, 1 \leq j \leq n$$

in the top row of the determinant. Thus (74) is equal to

$$\begin{aligned}(-1)^{n-w_1}&\frac{2^{\frac{n(n+1)}{2}}}{(n-1)!!}\cdot(C\cdot(-1)^{n-1}\cdot(n-1)!!)\\ &= C\cdot(-1)^{w_1+1}2^{\frac{n(n+1)}{2}}.\end{aligned}$$

times the coefficient of Δ^{n-1} in

$$(I+\Delta)^{d_0-1}(2I+\Delta)^{-(n+1-w_1)}\Delta^{n-w_1}.$$

Since, in the task of finding this coefficient, Δ plays a purely symbolic role, we might as well use a symbol which does not denote an operator, such as x. It will also be convenient to use the following notation, due to Richard Stanley.

Definition 4. The coefficient of x^j in the formal power series $f(x)$ on x is denoted by $[x^j](f(x))$.

Our task is to compute $[x^{n-1}]((1+x)^{d_0-1}(2+x)^{-(n+1-w_1)}x^{n-w_1}$. We have

$$
\begin{aligned}
[x^{n-1}]((1+x)^{d_0-1}(2+x)^{-(n+1-w_1)}x^{n-w_1}) &= [x^{w_1-1}]((1+x)^{d_0-1}(2+x)^{-(n+1-w_1)}) \\
&= [x^{w_1-1}]((2+x)^{w_1-1}\frac{(1+x)^{d_0-1}}{(2+x)^n}) \\
&= [(2y)^{w_1-1}]((2+2y)^{w_1-1}\frac{(1+2y)^{d_0-1}}{(2+2y)^n}) \\
&= 2^{-(w_1-1)}\cdot[y^{w_1-1}]((2+2y)^{w_1-1}\frac{(1+2y)^{d_0-1}}{(2+2y)^n}) \\
&= 2^{-(w_1-1)}\cdot 2^{w_1-1}2^{-n}\cdot[y^{w_1-1}]((1+y)^{w_1-1}\frac{(1+2y)^{d_0-1}}{(1+y)^n}) \\
&= 2^{-n}\cdot[y^{w_1-1}]((1+y)^{w_1-1}\frac{(1+2y)^{d_0-1}}{(1+y)^n})
\end{aligned}
$$

Now, for any formal power series $f(y)$ on y,

$$
\begin{aligned}
[y^{w_1-1}]((1+y)^{w_1-1}\cdot f(y)) &= \sum_{j=0}^{w_1-1}\binom{w_1-1}{j}[y^j](f(y)) \\
&= \sum_{j=0}^{w_1-1}\binom{(w_1-1-j)+j}{j}[y^j](f(y)) \\
&= \sum_{j=0}^{w_1-1}(([z^{w_1-1-j}](\frac{1}{(1-z)^{j+1}}))\cdot([y^j](f(y)))) \\
&= \sum_{j=0}^{w_1-1}(([z^{w_1-1}](\frac{z^j}{(1-z)^{j+1}}))\cdot([y^j](f(y)))) \\
&= \sum_{j=0}^{w_1-1}(([z^{w_1}](\frac{z^{j+1}}{(1-z)^{j+1}}))\cdot([y^{j+1}](y\cdot f(y)))) \\
&= [z^{w_1}]((\frac{z}{1-z})\cdot f(\frac{z}{1-z})) \\
&= [z^{w_1-1}]((\frac{1}{1-z})\cdot f(\frac{z}{1-z})).
\end{aligned} \tag{75}
$$

Therefore

$$
\begin{aligned}
((1+x)^{d_0-1}(2+x)^{-(n+1-w_1)}) &= 2^{-n}\cdot[y^{w_1-1}]((1+y)^{w_1-1}\frac{(1+2y)^{d_0-1}}{(1+y)^n})(\text{by } (3)) \\
&= 2^{-n}\cdot[z^{w_1-1}](\frac{1}{1-z}\cdot\frac{(1+\frac{2z}{1-z})^{d_0-1}}{(1+\frac{z}{1-z})^n}) \\
&= 2^{-n}\cdot[z^{w_1-1}](\frac{1}{1-z}\cdot(1+\frac{2z}{1-z})^{d_0-1}\cdot(1-z)^n) \\
&= 2^{-n}\cdot[z^{w_1-1}](\frac{1}{1-z}\cdot(\frac{1+z}{1-z})^{d_0-1}\cdot(1-z)^n) \\
&= 2^{-n}\cdot[z^{w_1-1}]((1+z)^{d_0-1}\cdot(1-z)^{(n-1)-(d_0-1)})
\end{aligned} \tag{76}
$$

An expression of this form is called a Krawtchouk polynomial ([8], p. 130).

We have just proven

Lemma 6. *The weighted number of matchings of an Aztec diamond of order n with a black hole at (d_0+1, w_1) and a white hole at $(1, w_1+d_1)$, $d_0, d_1 \geq 1$, is*

$$(-1)^{w_1+1} \cdot [z^{w_1-1}]((1+z)^{d_0-1} \cdot (1-z)^{(n-1)-(d_0-1)}) \cdot 2^{\frac{n(n-1)}{2}}.$$

We can now work on the general case. Every matching of an Aztec diamond of order n with a black hole at (w_0+d_0, w_1) and a white hole at (w_0, w_1+d_1) can be subdivided into one, and only one, pair of matchings of the following form.

1. The first item of the pair is a matching of a white-edged $n \times (w_1+d_1-1)$ Aztec rectangle with dents at $y_1, y_2, \cdots y_{w_1+d_1-2}$ and with a black hole at (w_0+d_0, w_1).
2. The second item of the pair is a matching of a white-edged $n \times (n-(w_1+d_1-1))$ Aztec rectangle with dents at $z_1, z_2, \cdots z_{n-(w_1+d_1-1)}$.

The numbers $y_1, y_2, \cdots y_{w_1+d_1-2}$, $z_1, z_2, \cdots z_{n-(w_1+d_1-1)}$ and w_0 are all distinct and cover all of the interval $\{1, 2, \cdots, n\}$. Thus every matching can be described as a partition of $\{1, 2, \cdots, n\} - \{w_0\}$ into two subsets, a matching of an Aztec rectangle with the first subset as its set of dents, and a matching of an Aztec rectangle with the second subset as its set of dents. As we stated in the previous section, we will weigh each matching T by a factor of $(-1)^{w(T)}$, where $w(T)$ is the sum of the number of edges of the form $((i-1, w_1+1), (i, w_1))$ or $((i, w_1+1), (i, w_1))$ for $1 \leq i < w_0+d_0$ and the number of edges of the form $((i, w_1+d_1), (i, w_1+d_1-1))$ or $((i, w_1+d_1), (i+1, w_1+d_1-1))$ for $1 \leq i < w_0$. Thus the weight $w(T)$ of a matching is equal to the weight of the matching of the $n \times (w_1+d_1-1)$ white-edged Aztec rectangle which it induces, plus the number of teeth of this Aztec rectangle whose indices are lower than w_0.

Hence the weighted number of matchings of an Aztec diamond of order n with a black hole at (w_0+d_0, w_1) and a white hole at (w_0, w_1+d_1) is equal to the sum of

$$(-1)^{d_1-1}(-1)^{t(w_0, y_1, \cdots, y_{w_1+d_1-2})} \cdot \frac{2^{\frac{(w_1+d_1-1)(w_1+d_1)}{2}}}{(w_1+d_1-2)!!} \cdot \begin{vmatrix} ((I+\Delta)^{w_0+d_0-1}((2I+\Delta)^{-d_1}\Delta^{d_1-1}(x^{j-1})))(1) \\ y_1^{j-1} \\ y_2^{j-1} \\ \cdots \\ y_{w_1+d_1-2}^{j-1} \end{vmatrix} \cdot \frac{2^{\frac{(n-(w_1+d_1-1))((n-(w_1+d_1-1))+1)}{2}}}{(n-(w_1+d_1-1)-1)!!} \left| z_i^{j-1} \right|_1^{n-(w_1+d_1-1)} \tag{77}$$

over all partitions of $\{1, 2, \cdots, n\} - \{w_0\}$ into two sets

$$\{y_1, y_2, \cdots y_{w_1+d_1-2}\}, \{z_1, z_2, \cdots z_{n-(w_1+d_1-1)}\},$$

where $y_1 < y_2 < \cdots < y_{w_1+d_1-2}$ and $z_1 < z_2 < \cdots < z_{n-(w_1+d_1-1)}$, and $t(w_0, y_1, \cdots, y_k)$ is equal to how many of $y_1, y_2, \cdots y_k$ are less than w_0.

We may dispose of the inconvenient $t(w_0, y_1, \cdots, y_{w_1+d_1-2})$ by expressing the same result as the sum of

$$(-1)^{d_1}\frac{2^{\frac{(w_1+d_1-1)(w_1+d_1)}{2}}}{(w_1+d_1-2)!!}\begin{vmatrix} 0 & ((I+\Delta)^{w_0+d_0-1}((2I+\Delta)^{-d_1}\Delta^{d_1-1}(x^{j-2})))(1) \\ \delta_{v_1,w_0} & v_1^{j-2} \\ \delta_{v_2,w_0} & v_2^{j-2} \\ \cdots & \\ \delta_{v_{w_1+d_1-1},w_0} & v_{w_1+d_1-1}^{j-2} \end{vmatrix} \cdot \frac{2^{\frac{(n-(w_1+d_1-1))((n-(w_1+d_1-1))+1)}{2}}}{(n-(w_1+d_1-1)-1)!!}\left|z_i^{j-1}\right|_1^{n-(w_1+d_1-1)} \tag{78}$$

over all partitions of $\{1, 2, \cdots, n\}$ into two sets $\{v_1, \cdots, v_{w_1+d_1-1}\}$, $z_1, \cdots z_{n-(w_1+d_1-1)}$, where $v_1 < v_2 < \cdots < v_{w_1+d_1-1}$ and $z_1 < z_2 < \cdots < z_{n-(w_1+d_1-1)}$, and where $\delta_{i,j}$ is equal to 1 for $i = j$, 0 for $i \neq j$.

The following lemma follows immediately from Laplace's development of a determinant ([12], pp. 22-25).

Lemma 7. *Let us have a determinant* $|b_{i,j}|_1^{m_1+m_2}$*, where* $b_{i,j} = c_{i,j}$ *for* $j \leq m_1$*,* $b_{i,j} = (-1)^{i-1}d_{i,j-m_1}$ *for* $j > m_1$*. Then*

$$|b_{i,j}|_1^{m_1+m_2} = (-1)^{\lfloor\frac{m_1+m_2}{2}-\frac{m_1}{2}\rfloor}\cdot\sum|c_{x_i,j}|_1^{m_1}\cdot|d_{y_i,j}|_1^{m_2} \tag{79}$$

where the sum is over all partitions of $\{1, 2, \cdots, m_1+m_2\}$ *into two sets* $\{x_1, x_2, \cdots, x_{m_1}\}$*,* $\{y_1, y_2, \cdots, y_{m_2}\}$*, where* $x_1 < x_2 < \cdots < x_{m_1}$ *and* $y_1 < y_2 < \cdots < y_{m_2}$*, and where* $\lfloor z\rfloor$ *is the largest integer less than or equal to* z*.*

We can now express our sum over partitions as a determinant. The weighted number of matchings of an Aztec diamond with two holes is

$$(-1)^{d_1}\frac{2^{\frac{(w_1+d_1-1)(w_1+d_1)}{2}}}{(w_1+d_1-2)!!}\frac{2^{\frac{(n-(w_1+d_1-1))((n-(w_1+d_1-1))+1)}{2}}}{(n-(w_1+d_1-1)-1)!!}(-1)^{\lfloor\frac{n+1}{2}\rfloor-\lfloor\frac{w_1+d_1}{2}\rfloor}|d_{i,j}|_1^{n+1}, \tag{80}$$

where

- $d_{i,1} = 0$ for all $1 \leq i \leq n+1$, $i \neq w_0+1$,
- $d_{w_0+1,1} = 1$,
- $d_{1,j} = ((I+\Delta)^{w_0+d_0-1}((2I+\Delta)^{-d_1}\Delta^{d_1-1}(x^{j-2})))(1)$ for all $1 < j \leq w_1+d_1$,
- $d_{1,j} = 0$ for all $w_1+d_1 < j \leq n+1$,
- $d_{i,j} = (i-1)^{j-2}$ for $i > 1$, $1 < j \leq w_1+d_1$,
- $d_{i,j} = (-1)^{i-1}(i-1)^{j-(w_1+d_1+1)}$ for $i > 1$, $j > w_1+d_1$

The task ahead is to compute the determinant $|d_{i,j}|_1^{n+1}$. For convenience, we will refer to it as $D(w_0, d_0, w_1, d_1)$. From (80) and Lemma 6, it follows that

$$D(1, d_0, w_1, d_1) = ((-1)^{d_1}\frac{2^{\frac{(w_1+d_1-1)(w_1+d_1)}{2}}}{(w_1+d_1-2)!!}\frac{2^{\frac{(n-(w_1+d_1-1))((n-(w_1+d_1-1))+1)}{2}}}{(n-(w_1+d_1-1)-1)!!}(-1)^{\lfloor\frac{n+1}{2}\rfloor-\lfloor\frac{w_1+d_1}{2}\rfloor})^{-1}\cdot$$
$$(-1)^{w_1+1}\cdot[z^{w_1-1}]((1+z)^{d_0-1}\cdot(1-z)^{(n-1)-(d_0-1)})\cdot 2^{\frac{n(n-1)}{2}}.$$

We will reduce the general case to the special case $w_0 = 0$ by expressing $D(w_0, d_0, w_1, d_1) - D(w_0 - 1, d_0, w_1, d_1)$, as the product of $D(1, d_0 + w_0 - 1, w_1, d_1)$ times something else.

If we take the determinant $|d_{i,j}|_1^{n+1}$ for $D(w_0 - 1, d_0, w_1, d_1)$, and add one to the bases of all powers, we obtain

$$D(w_0 - 1, d_0, w_1, d_1) = |g_{i,j}|_1^{n+1} \tag{81}$$

where

- $g_{i,1} = 0$ for all $1 \le i \le n+1$, $i \ne w_0$;
- $g_{w_0,1} = 1$;
- $g_{1,j} = ((I+\Delta)^{w_0+d_0-1}((2I+\Delta)^{-d_1}\Delta^{d_1-1}(x^{j-2})))(1)$ for all $1 < j \le w_1 + d_1$; notice how we have raised the exponent of $(I+\Delta)$ from $w_0 + d_0 - 2$ to $w_0 + d_0 - 1$;
- $g_{1,j} = 0$ for all $w_1 + d_1 < j \le n+1$;
- $g_{i,j} = i^{j-2}$ for $i > 1$, $1 < j \le w_1 + d_1$,
- $g_{i,j} = (-1)^{i-1} i^{j-(w_1+d_1+1)}$ for $i > 1$, $j > w_1 + d_1$,

The n-tuple $|g_{n+1,j}|_2^{n+1}$ is a linear combination of the n-tuples $|g_{i,j}|_2^{n+1}$, $2 \le i < n+1$ and of the n-tuple $\{1\}_2^{n+1}$, which is what $g_{1,j}$ would be if the pattern for $\{g_{n,j}\}_2^{n+1}$, $\{g_{n-1,j}\}_2^{n+1}$, $\ldots$ $\{g_{2,j}\}_2^{n+1}$ were continued.

Lemma 8. *Let a_k be the coefficient of x^k in $((x-1)^{w_1+d_1-1}(x+1)^{n-(w_1+d_1)+1})$. Then*

$$\sum_{k=0}^{n} a_k (k+1)^{j-2} = 0 \text{ for } 1 < j \le w_1 + d_1, \tag{82}$$

$$\sum_{k=0}^{n} a_k (-1)^k (k+1)^{j-(w_1+d_1+1)} = 0 \text{ for } w_1 + d_1 + 1 \le j \le n+1, \tag{83}$$

Proof.

$$\Delta^{w_1+d_1-1} x^{j-2} = 0 \text{ for } 1 < j \le w_1 + d_1 \tag{84}$$

implies

$$\Delta^{w_1+d_1-1}(\Delta + 2I)^{n-(w_1+d_1)+1} x^{j-2} = 0 \text{ for } 1 < j \le w_1 + d_1 \tag{85}$$

If we take the value at $x = 1$, we obtain

$$\sum_{k=0}^{n} a_k (k+1)^{j-2} = 0 \text{ for } 1 < j \le w_1 + d_1. \tag{86}$$

Similarly,

$$\Delta^{n-(w_1+d_1)+1} x^{j-(w_1+d_1+1)} = 0 \text{ for } w_1 + d_1 + 1 \le j \le n+1 \tag{87}$$

implies

$$(\Delta + 2I)^{w_1+d_1-1}(-\Delta)^{n-(w_1+d_1)+1} x^{j-(w_1+d_1+1)} = 0 \text{ for } w_1 + d_1 + 1 \le j \le n+1. \tag{88}$$

If we let H be the operator taking x^j to $(x+1)^j$, we can write

$$(I+H)^{w_1+d_1-1}(I-H)^{n-(w_1+d_1)+1} x^{j-(w_1+d_1+1)} = 0 \text{ for } w_1 + d_1 + 1 \le j \le n+1 \tag{89}$$

Clearly the coefficient of H^k in

$$(I+H)^{w_1+d_1-1}(I-H)^{n-(w_1+d_1)+1}$$

is equal to $(-1)^k$ times the coefficient of H^k in

$$(I-H)^{w_1+d_1-1}(I+H)^{n-(w_1+d_1)+1},$$

which is equal to $(-1)^{w_1+d_1-1}$ times the coefficient of H^k in

$$(H-I)^{w_1+d_1-1}(H+I)^{n-(w_1+d_1)+1},$$

that is, a_k. Hence

$$\begin{aligned}(-1)^{w_1+d_1-1}\sum_{k=0}^{n}(-1)^k a_k(k+1)^{j-(w_1+d_1+1)} &= \sum_{k=0}^{n}((-1)^{w_1+d_1-1}(-1)^k a_k)(k+1)^{j-(w_1+d_1+1)} \\ &= (I+H)^{w_1+d_1-1}(I-H)^{n-(w_1+d_1)+1}x^{j-(w_1+d_1+1)}(1) \\ &= (\Delta+2I)^{w_1+d_1-1}(-\Delta)^{n-(w_1+d_1)+1}x^{j-(w_1+d_1+1)} \\ &= (\Delta+2I)^{w_1+d_1-1}\Delta^{n-(w_1+d_1)+1}x^{j-(w_1+d_1+1)} \\ &= 0 \text{ for } w_1+d_1+1 \le j \le n+1.\end{aligned}$$

Therefore

$$\sum_{k=0}^{n} a_k(-1)^k(k+1)^{j-(w_1+d_1+1)} = 0 \text{ for } w_1+d_1+1 \le j \le n+1.$$

□

Hence, if we add a_n times the nth row, a_{n-1} times the $(n-1)$th row, … a_2 times the second row to the bottom row of $|g_{i,j}|_1^{n+1}$, we obtain the row $\{h_j\}_1^{n+1}$, where

- $h_0 = [x^{w_0-1}]((x-1)^{w_1+d_1-1}(x+1)^{n+1-(w_1+d_1)})$,
- $h_j = (-1)\cdot(-1)^{w_1+d_1-1}$ for $1 < j \le w_1+d_1$,
- $h_j = (-1)\cdot(-1)^{w_1+d_1-1}$ for $j > w_1+d_1$.

After switching signs and shifting the bottom row of $D(w_0-1,d_0,w_1,d_1)$ (now $\{h_j\}_1^{n+1}$) to the second-to-topmost place, we obtain

$$D(w_0-1,d_0,w_1,d_1) = (-1)\cdot(-1)^{w_1+d_1-1}\cdot(-1)^{n-1}\cdot|k_{i,j}|_1^{n+1}, \tag{90}$$

where

- $k_{i,1} = 0$ for all $1 \le i \le n+1$, $i \ne w_0+1$, $i \ne 2$;
- $k_{2,1} = (-1)\cdot(-1)^{w_1+d_1-1}\cdot[x^{w_0-1}]((x-1)^{w_1+d_1-1}(x+1)^{n+1-(w_1+d_1)})$;
- $k_{w_0+1,1} = 1$;
- $k_{1,j} = ((I+\Delta)^{w_0+d_0-1}((2I+\Delta)^{-d_1}\Delta^{d_1-1}(x^{j-2})))(1)$ for all $1 < j \le w_1+d_1$;
- $k_{1,j} = 0$ for $w_1+d_1+1 \le j \le n+1$;
- $k_{i,j} = (i-1)^{j-2}$ for $i > 1$, $1 < j \le w_1+d_1$;
- $k_{i,j} = (-1)^i(i-1)^{j-(w_1+d_1+1)}$ for $i > 1$, $j \ge w_1+d_1+1$.

We multiply the $n-(w_1+d_1)+1$ rightmost columns by (-1), obtaining

$$D(w_0-1,d_0,w_1,d_1) = |l_{i,j}|_1^{n+1}, \tag{91}$$

where

- $l_{i,1} = 0$ for all $1 \leq i \leq n+1$, $i \neq w_0+1$, $i \neq 2$;
- $l_{2,1} = (-1) \cdot (-1)^{w_1+d_1-1} \cdot [x^{w_0-1}]((x-1)^{w_1+d_1-1}(x+1)^{n+1-(w_1+d_1)})$;
- $l_{w_0+1,1} = 1$;
- $l_{1,j} = ((I+\Delta)^{w_0+d_0-1}((2I+\Delta)^{-d_1}\Delta^{d_1-1}(x^{j-2})))(1)$ for all $1 < j \leq w_1+d_1$;
- $l_{1,j} = 0$ for $w_1+d_1+1 \leq j \leq n+1$;
- $l_{i,j} = (i-1)^{j-2}$ for $i > 1$, $1 < j \leq w_1+d_1$;
- $l_{i,j} = (-1)^{i-1}(i-1)^{j-(w_1+d_1+1)}$ for $i > 1$, $j \geq w_1+d_1+1$.

Therefore

$$D(w_0,d_0,w_1,d_1) - D(w_0-1,d_0,w_1,d_1) = |r_{i,j}|_1^{n+1}, \tag{92}$$

where

- $r_{i,1} = 0$ for all $1 \leq i \leq n+1$, $i \neq 2$;
- $r_{2,1} = (-1)^{w_1+d_1-1} \cdot [x^{w_0-1}]((x-1)^{w_1+d_1-1}(x+1)^{n+1-(w_1+d_1)})$;
- $r_{1,j} = ((I+\Delta)^{w_0+d_0-1}((2I+\Delta)^{-d_1}\Delta^{d_1-1}(x^{j-2})))(1)$ for all $1 < j \leq w_1+d_1$;
- $r_{1,j} = 0$ for $w_1+d_1+1 \leq j \leq n+1$;
- $r_{i,j} = (i-1)^{j-2}$ for $i > 1$, $1 < j \leq w_1+d_1$;
- $r_{i,j} = (-1)^{i-1}(i-1)^{j-(w_1+d_1+1)}$ for $i > 1$, $j \geq w_1+d_1+1$.

This is equal to

$$\begin{aligned} r_{2,1} \cdot D(1,w_0+d_0-1,w_1,d_1) = {} & (-1)^{w_1+d_1-1} \cdot [x^{w_0-1}]((x-1)^{w_1+d_1-1}(x+1)^{n+1-(w_1+d_1)}) \cdot \\ & D(1,w_0+d_0-1,w_1,d_1) \\ & = [x^{w_0-1}]((1-x)^{w_1+d_1-1}(1+x)^{(n-(w_1+d_1-1))}) \cdot \\ & D(1,w_0+d_0-1,w_1,d_1) \end{aligned} \tag{93}$$

Therefore

$$\begin{aligned} D(w_0,d_0,w_1,d_1) &= (\sum_{j=1}^{w_0-1} D(j+1,d_0,w_1,d_1) - D(j,d_0,w_1,d_1)) \\ &+ D(1,d_0,w_1,d_1) \\ &= \sum_{j=1}^{w_0-1} ([x^j](1-x)^{w_1+d_1-1}(1+x)^{n-(w_1+d_1-1)}) \cdot D(1,j+d_0,w_1,d_1) \\ &+ D(1,d_0,w_1,d_1) \\ &= \sum_{j=0}^{w_0-1} ([x^j](1-x)^{w_1+d_1-1}(1+x)^{n-(w_1+d_1-1)}) \cdot D(1,j+d_0,w_1,d_1) \end{aligned} \tag{94}$$

By (80), it follows that the weighted number of matchings $\sum(-1)^{w(T)}$ of an Aztec diamond of order n with a black hole at (w_0+d_0, w_1) and a white hole at (w_0, w_1+d_1) is equal to

$$
\begin{aligned}
&(-1)^{d_1}\frac{2^{\frac{(w_1+d_1-1)(w_1+d_1)}{2}}}{(w_1+d_1-2)!!}\frac{2^{\frac{(n-(w_1+d_1-1))((n-(w_1+d_1-1))+1)}{2}}}{(n-(w_1+d_1-1)-1)!!}(-1)^{\lfloor\frac{n+1}{2}\rfloor-\lfloor\frac{w_1+d_1}{2}\rfloor}\cdot\\
&\sum_{j=0}^{w_0-1}([x^j](1-x)^{w_1+d_1-1}(1+x)^{n-(w_1+d_1-1)})\cdot D(1,j+d_0,w_1,d_1)\\
&=\sum_{j=0}^{w_0-1}(([x^j](1-x)^{w_1+d_1-1}(1+x)^{n-(w_1+d_1-1)})\cdot\\
&((-1)^{d_1}\frac{2^{\frac{(w_1+d_1-1)(w_1+d_1)}{2}}}{(w_1+d_1-2)!!}\frac{2^{\frac{(n-(w_1+d_1-1))((n-(w_1+d_1-1))+1)}{2}}}{(n-(w_1+d_1-1)-1)!!}\cdot\\
&(-1)^{\lfloor\frac{n+1}{2}\rfloor-\lfloor\frac{w_1+d_1}{2}\rfloor}\cdot D(1,j+d_0,w_1,d_1))
\end{aligned}
$$

The term within parentheses including $D(1,j+d_0,w_1,d_1)$ is equal to the number of matchings of an Aztec diamond of order n with a black hole at $(j+d_0+1,w_1)$ and a white hole at $(1,w_1+d_1)$. By Lemma 6, this number is equal to

$$(-1)^{w_1+1}\cdot 2^{\frac{n(n-1)}{2}}\cdot[z^{w_1-1}]((1+z)^{j+d_0-1}\cdot(1-z)^{(n-1)-(j+d_0-1)})$$

Hence $\sum(-1)^{w(T)}$ is equal to

$$
\begin{aligned}
(-1)^{w_1+1}\cdot 2^{-n}\cdot\sum_{j=0}^{w_0-1}&([x^j]((1-x)^{w_1+d_1-1}(1+x)^{n-(w_1+d_1-1)})\cdot\\
&[z^{w_1-1}]((1+z)^{j+d_0-1}\cdot(1-z)^{(n-1)-(j+d_0-1)})),
\end{aligned}
$$

From this and from (2.4) the result we have sought follows immediately.

Proposition 9. *The entry in the inverse of the Kasteleyn matrix of an Aztec diamond of order n corresponding to a black square at (w_0+d_0,w_1) and a white square at (w_0,w_1+d_1), $d_0,d_1>0$, is*

$$
\begin{aligned}
(-1)^{d_0+d_1+w_1}\cdot 2^{-n}\cdot\sum_{j=0}^{w_0-1}&([x^j]((1-x)^{w_1+d_1-1}(1+x)^{n-(w_1+d_1-1)})\cdot\\
&[z^{w_1-1}]((1+z)^{j+d_0-1}\cdot(1-z)^{(n-1)-(j+d_0-1)})),
\end{aligned}
\tag{95}
$$

where $[x^j](p(x))$ is the coefficient of x^j in the polynomial $p(x)$. (Alternatively, this can be called the value of the coupling function of the Aztec diamond of order n at the black hole (w_0+d_0,w_1) and the white hole (w_0,w_1+d_1).)

In order to deal with the cases $d_0\leq 0$ and $d_1\leq 0$, we merely need to flip the Aztec diamond so as to make $d_0,d_1>0$ and compute the weighted number of tilings in the manner we have described. Of course, we have to account for the fact that the weighting has to be computed differently. For $d_0\leq 0$, we also have to express $D(w_0,d_0,w_1,d_1)$ as

$$-\sum_{j=w_0}^{n}D(j+1,d_0,w_1,d_1)-D(j,d_0,w_1,d_1)$$

and not as

$$D(1, d_0, w_1, d_1) + \sum_{j=1}^{w_0-1} D(j+1, d_0, w_1, d_1) - D(j, d_0, w_1, d_1))$$

as we did in (94). These are the only two details worth mention in the otherwise trivial derivation of the following result from Proposition 9.

Corollary 10. *The coupling function of the Aztec diamond of order n at the black square $(w_0 + d_0, w_1)$ and the white square $(w_0, w_1 + d_1)$ is*

$$\begin{aligned}(-1)^{d_0+d_1+w_1} \cdot 2^{-n} \cdot \sum_{j=0}^{w_0-1} ([x^j]((1-x)^{w_1+d_1-1}(1+x)^{n-(w_1+d_1-1)}) \cdot \\ [z^{w_1-1}]((1+z)^{j+d_0-1} \cdot (1-z)^{(n-1)-(j+d_0-1)}))\end{aligned} \tag{96}$$

for $d_0 > 0$,

$$\begin{aligned}(-1)^{d_0+d_1+w_1} \cdot 2^{-n} \cdot (-\sum_{j=w_0}^{n} ([x^j]((1-x)^{w_1+d_1-1}(1+x)^{n-(w_1+d_1-1)}) \cdot \\ [z^{w_1-1}]((1+z)^{j+d_0-1} \cdot (1-z)^{(n-1)-(j+d_0-1)})))\end{aligned} \tag{97}$$

for $d_0 \leq 0$.

When we take a minor of the inverse Kasteleyn matrix, the factors $(-1)^{d_0+d_1+w_1}$, multiplied, give the same product in every term of the expression of the minor in a form such as $\sum_\pi \operatorname{sgn}(\pi) \prod_{i=1}^k c_{i,\pi(i)}$. Thus we can leave them out, and our main result follows.

Theorem 11. *The probability of a pattern covering white squares $v_1, v_2, \cdots v_k$ and black squares $w_1, w_2, \cdots w_k$ of an Aztec diamond of order n is equal to the absolute value of*

$$|c(v_i, w_j)|_{i,j=1,2,\cdots k}\,.$$

The coupling function $c(v, w)$ *at white square v and black square w is*

$$2^{-n} \sum_{j=0}^{x_i-1} \operatorname{kr}(j, n, y_i) \operatorname{kr}(y\prime_i - 1, n-1, n-(j + x\prime_i - x_i))$$

for $x\prime_i > x_i$ and

$$-2^{-n} \sum_{j=x_i}^{n} \operatorname{kr}(j, n, y_i) \operatorname{kr}(y\prime_i - 1, n-1, n-(j + x\prime_i - x_i))$$

for $x\prime_i \leq x_i$, where (x_i, y_i) and $(x\prime_i, y\prime_i)$ are the coordinates of v and w, respectively, in the coordinate system in figure 17, and the Krawtchouk polynomial $\operatorname{kr}(a, b, c)$ *is the coefficient of x^a in $(1-x)^b \cdot (1+x)^c$.*

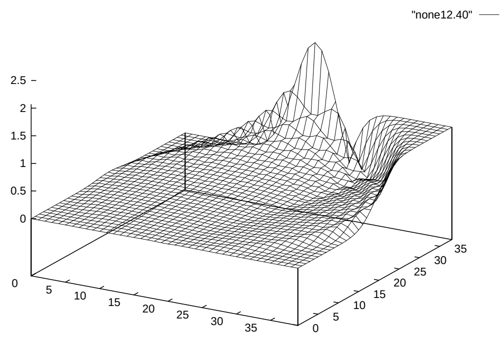

FIGURE 19. Inverse Kasteleyn matrix entries as a function of w_0, w_1 for $d_0 = 1$, $d_1 = 2$, $n = 40$

4. In perspective

Proposition 9 is valuable in itself, in that it gives us an efficient algorithm for computing an arbitrary entry in the inverse Kasteleyn matrix of the Aztec diamond. Figures 19 to 24 show the absolute value of the entry as a function of w_0 and w_1, for fixed d_0 and d_1. As we showed in section 2, given a pattern consisting of k vertices, we can compute the probability of its occurence at any point in a Aztec diamond of given order by computing $(k/2)^2$ entries of the inverse Kasteleyn matrix. Thus, for a fixed pattern, the time required for computing its probability is equal to a constant times the time required for computing an entry of the inverse Kasteleyn matrix using Proposition 9. Whether computation time grows quadratically on the order of the Aztec diamond, or somewhat faster, depends on whether multiplying integers is assumed to take constant time. What is clear is that we now have an algorithm that is much more efficient than computing the entries of an inverse Kasteleyn matrix by actually inverting the matrix or computing minors.

Proposition 9 gives us an expression that is more closed than an entry in the inverse of a Kasteleyn matrix. What do we mean by this? There are few tools available that would allow us to obtain asymptotic expressions for a sequence of entries in a sequence of inverses of arbitrary matrices. For finding the asymptotics of sums such as (95), however, there are many well-developed analytical techniques. At the time of this writing, Henry Cohn is working on some minor problems involved in applying the saddle-point technique to the asymptotics of (95). Once he superates these difficulties (something that seems to be about to happen), the goals set in the introduction will have been achieved completely.

5. Acknowledgements

The author would like to thank Matthew Blum for converting the output of the urban-renewal program `ren.c` to figures 3 to 13. He would also like to thank James Propp and Ira Gessel for their support, and Henry Cohn, in advance, for finishing his work on the asymptotics of the herein described results and for writing the still unwritten continuation to this paper.

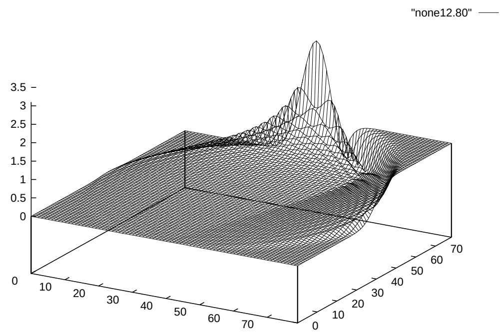

FIGURE 20. Inverse Kasteleyn matrix entries as a function of w_0, w_1 for $d_0 = 1$, $d_1 = 2$, $n = 80$

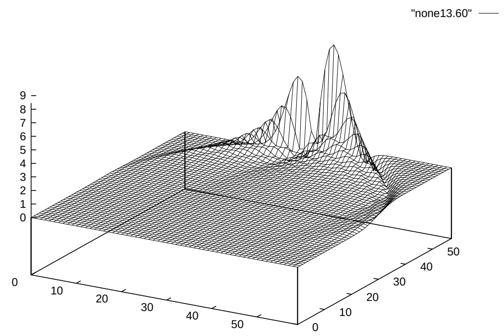

FIGURE 21. Inverse Kasteleyn matrix entries as a function of w_0, w_1 for $d_0 = 1$, $d_1 = 3$, $n = 60$

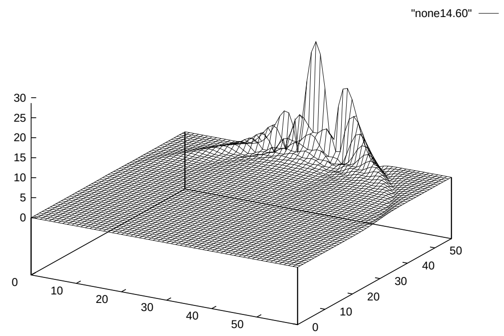

FIGURE 22. Inverse Kasteleyn matrix entries as a function of w_0, w_1 for $d_0 = 1$, $d_1 = 4$, $n = 60$

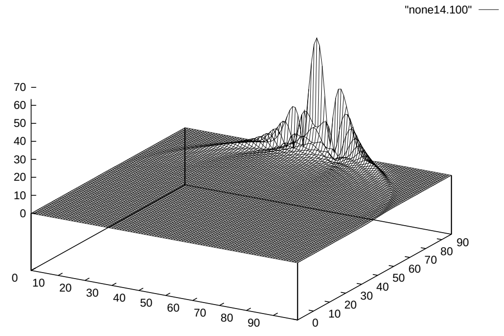

FIGURE 23. Inverse Kasteleyn matrix entries as a function of w_0, w_1 for $d_0 = 1$, $d_1 = 4$, $n = 100$

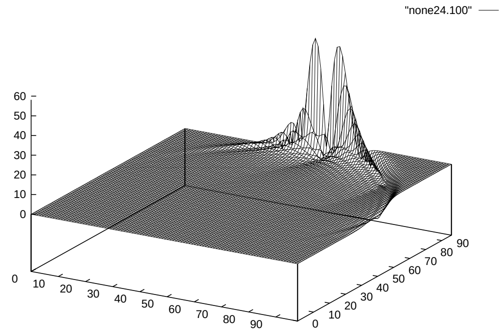

FIGURE 24. Inverse Kasteleyn matrix entries as a function of w_0, w_1 for $d_0 = 2$, $d_1 = 4$, $n = 100$

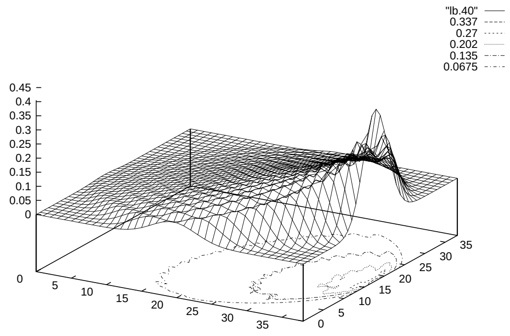

FIGURE 25. Probability of occurence of the shape in figure 27 as a function of position, for the Aztec diamond of side 40.

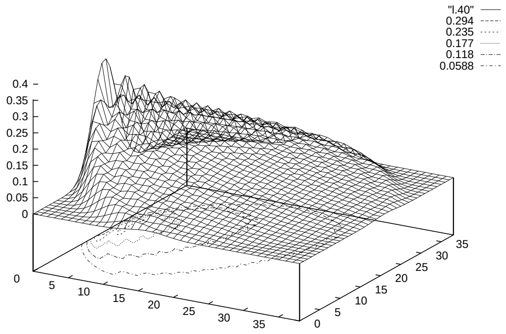

FIGURE 26. Probability of occurence of the shape in figure 28 as a function of position, for the Aztec diamond of side 40.

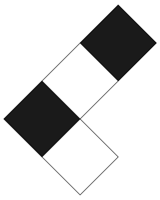

FIGURE 27. L-shape

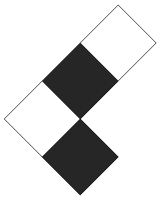

FIGURE 28. L-shape

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