Math 431, Assignment #6: Solutions

(due 3/29/01)

1. $E((X-t)^2) = E(X^2 - 2tX + t^2) = E(X^2) - 2tE(X) + t^2$. Setting the derivative equal to zero, we find that this function of t has a critical point at $t = E(X)$, and for this value of t, $E((X-t)^2)$ takes the value $\mathrm{Var}(t)$. To see that this unique critical point is the global minimum, we use the second derivative test: the second derivative of $E((X-t)^2)$, evaluated at $t = E(X)$, is 2, which is positive. Alternatively, note that the graph of $f(t) = E(X^2) - 2tE(X) + t^2$ is just a parabola, and that the coefficient of t^2 is positive.

2. First solution: For all k between 0 and n, the probability of the event $\{X = k\}$ is $\binom{n}{k} p^k q^{n-k}$, where $q = 1 - p$. Hence the expected value of α^X is $\sum_{k=0}^{n} \binom{n}{k} p^k q^{n-k} \alpha^k$, which can be rewritten as $\sum_{k=0}^{n} \binom{n}{k} (p\alpha)^k q^{n-k}$. By the binomial theorem, this is is $(p\alpha + q)^n$.

   Second solution: Write $X = Y_1 + ... + Y_n$, where $Y_1, ... Y_n$ are independently identically distributed Bernoulli random variables, each with parameter p. Then $\alpha^X = \alpha^{Y_1} \cdots \alpha^{Y_n}$. Since the random variables $\alpha^{Y_1}, ..., \alpha^{Y_n}$, like the random variables $Y_1, ..., Y_n$, are independent of one another, we see that $E(\alpha^X) = E(\alpha^{Y_1}) E(\alpha^{Y_2}) \cdots E(\alpha^{Y_n})$. Since each Y_i takes the value $\alpha^1 = \alpha$ with probability p and the value $\alpha^0 = 1$ with probability q, the expected value of α^{Y_i} is $\alpha p + q$, and so we have $E(\alpha^X) = (\alpha p + q)^n$.

3. V is a negative binomial random variable with parameters $p = 1/6$, $r = 3$. W is a geometric random variable with parameter $p = 1/6$.

   (a) Method 1:

$$\begin{aligned}
P(V = 6 \mid V > 5) &= P(V = 6 \text{ and } V > 5)/P(V > 5) \\
&= P(V = 6)/P(V > 5) \\
&= P(V = 6)/(1 - P(V = 3) - P(V = 4) - P(V = 5)).
\end{aligned}$$

Applying the formula $P(V = k) = \binom{k-1}{3-1} p^3 q^{k-3}$ with $p = 1/6$ and $q = 5/6$, and with k taking the values 5, 6, 7, and 8, we substitute to get $P(V = 6 \mid V > 5) = 1/36$.

Method 2: As before, $P(V = 6 \mid V > 5) = P(V = 6)/P(V > 5)$. $P(V = 6) = \binom{5}{2} p^3 q^3 = 10 \times 5^3/6^6$. The event $V > 5$ can happen in three different ways, according to whether there are 0, 1, or 2 successes in the first 5 trials. Hence $P(V > 5) = \binom{5}{0} p^0 q^5 + \binom{5}{1} p^1 q^4 + \binom{5}{2} p^2 q^3 = (5^5 + 5 \times 5^4 + 10 \times 5^3)/6^5$. So, cancelling out a factor of $5^3/6^5$, we get $P(V = 6 \mid V > 5) = P(V = 6)/P(V > 5) = (10/6)/(5^2 + 5 \times 5 + 10) = (10/6)/60 = 1/36$.

(b) This time, Method 1 would be very impractical, since it would require writing $P(V > 40)$ as $1 - \sum_{k=3}^{40} P(V = k)$. But with Method 2, we have $P(V > 40) = \binom{40}{0} p^0 q^{40} + \binom{40}{1} p^1 q^{39} + \binom{40}{2} p^2 q^{38} = (5^{40} + 40 \times 5^{39} + 780 \times 5^{38})/6^{40}$. Since $P(V = 41) = \binom{40}{2} p^3 q^{38} = 780 \times 5^{38}/6^{41}$, we can divide as before to get $P(V = 41 \mid V > 40) = P(V = 41)/P(V > 40) = (780/6)/(5^2 + 40 \times 5 + 780) = 26/201 = .129353....$

(c) The memorylessness of a geometric random variable tells us that $P(W = 6 \mid W > 5) = P(W = 1) = P(\text{success on 1st trial}) = 1/6$.

(d) The memorylessness of a geometric random variable tells us that $P(W = 41 \mid W > 40) = P(W = 1) = P(\text{success on 1st trial}) = 1/6$.

(e) In both (a) and (b), we know that we have not yet seen the third success, but we do not know whether the first two successes have occurred yet. But in (a), since there have been only five trials, it is very unlikely that there have already been two successes, and therefore very unlikely that the third success will happen on the next trial. In (b), it is fairly likely that the first two successes have already occurred, and therefore the probability is much closer to 1/6 that the third success will occur on the next trial.

If we consider $P(V = 1001 \mid V > 1000)$, we realize that after 1000 trials, it is extremely probably that the first two successes have already occurred, and therefore we would have $P(V = 1001 \mid V > 1000)$ extremely close to 1/6.

(f) We can infer from the memorylessness of geometric random variables that these two conditional probabilities are equal.

4. $P(X = k) = \binom{k-1}{r-1} p^r (1-p)^{k-r}$, so $\log P(X = k) = \log \binom{k-1}{r-1} + r \log p + (k-r) \log(1-p)$. This has a critical point where $0 = \frac{r}{p} - \frac{k-r}{1-p}$, which happens exactly when $p = r/k$. This local maximum is in fact a global maximum, since there is only one critical point and since the value achieved there is greater than the value at the endpoints (leaving aside the easy case $k = r$, which can be handled separately).

5. (a) $1 - e^{-3.5} - 3.5e^{-3.5} = 1 - 4.5e^{-3.5} \approx .864$

   (b) $4.5e^{-3.5} \approx .136$

   Since each flight has a small probability of crashing it seems reasonable to suppose that the number of crashes is approximately Poisson distributed.

6. (a) The probability that an arbitrary couple were both born on April 30 is $(1/365)^2$, assuming independence and an equal chance of having been born on any given date. Hence, the number of such couples is approximately Poisson with mean $80,000/(365)^2 \approx .6$. Therefore, the probability that at least one pair were both born on this date is approximately $1 - e^{-.6} \approx .451$.

   (b) The probability that an arbitrary couple were both born on the same day of the year is $1/365$. Hence, the number of such couples is approximately Poisson with mean $80,000/365 \approx 219.18$. Therefore, the probability that at least one pair were both born on the same date is $1 - e^{-219.18} \approx 1$.

   Note: For part (b), the "true" probability of there being no two partners born on the same day as each other is $(1 - \frac{1}{365})^{80,000}$, whose (natural) logarithm is $80,000 \ln(1 - \frac{1}{365}) \approx -219.48$. Note how close $1 - e^{-219.48}$ is to close $1 - e^{-219.18}$. So we see that the Poisson approximation is a good assumption for part (b), even though pn is far from 1.

7. $\frac{1}{2}e^{-3} + \frac{1}{2}e^{-4.2} \approx .0324$

8. (a) $1 - e^{-3} - 3e^{-3} - e^{-3}\frac{3^2}{2} = 1 - \frac{17}{2}e^{-3} \approx .577$

(b) $P(X \geq 3 \mid X \geq 1) = \frac{P(X\geq 3)}{P(X\geq 1)} = \frac{1-\frac{17}{2}e^{-3}}{1-e^{-3}} \approx .607$

9. $\log P(X = k) = -\lambda + k \log \lambda - \log(k!)$. We have $0 = \frac{\partial}{\partial \lambda} \log P(X = k) = -1 + \frac{k}{\lambda}$ exactly when $\lambda = k$.

10. $P(X = n + k \mid X > n) = \frac{P(X=n+k)}{P(X>n)} = \frac{p(1-p)^{n+k-1}}{(1-p)^n} = p(1-p)^{k-1}$. Intuitively, if the first n trials are failures, then at that point it is as if we are beginning anew.