Math 431, Assignment #4: Solutions

(due 3/8/01)

1. (a) The maximum of two numbers, each of which is between 1 and 6, must be between 1 and 6. To compute the number of pairs (i, j) with $\max(i, j) = 1$, we use the inclusion-exclusion principle. For each fixed m, the number of pairs (m, j) with $j \leq m$ is m and the number of pairs (i, m) with $i \leq m$ is m; the intersection of these two sets of outcomes contains the single outcome (m, m). Hence the number of pairs (i, j) with $\max(i, j) = m$ is $m + m - 1$, or $2m - 1$, and the probability that the maximum of the values of the two rolls is m is $(2m - 1)/36$. More specifically, $P(X = 1) = 1/36$, $P(X = 2) = 3/36$, $P(X = 3) = 5/36$, $P(X = 4) = 7/36$, $P(X = 5) = 9/36$, and $P(X = 6) = 11/36$, where X denote the maximum value to appear on the two rolls. (Another way to find these probabilities is to draw a 6-by-6 grid with rows indexed by i and columns indexed by j, with the number $\max(i, j)$ written in the i, jth box, and then just count the number of times the number m occurs in the grid.) The expected value of X is $(1)(1/36) + (2)(3/36) + (3)(5/36) + (4)(7/36) + (5)(9/36) + (6)(11/36) = 161/36$.

   (b) The same reasoning tells us that the number of pairs (i, j) with $\min(i, j) = m$ is $(7 - m) + (7 - m) - 1 = 13 - 2m$. So, letting Y denote the minimum value to appear on the two rolls, we have $P(Y = 1) = 11/36$, $P(Y = 2) = 9/36$, $P(Y = 3) = 7/36$, $P(Y = 4) = 5/36$, $P(Y = 5) = 3/36$, and $P(Y = 6) = 1/36$. The expected value of Y is $(1)(11/36)+(2)(9/36)+(3)(7/36)+(4)(5/36)+(5)(3/36)+(6)(1/36) = 91/36$. To check this, note that this gives $E(X+Y) = 161/36+91/36 = 252/36 = 7$. This makes sense, because the random variable $X + Y$ is the SAME RANDOM VARIABLE (that is, the same function) as the sum of the numbers seen on the two dice. And we know what the expected value of that sum is, thanks to part (c), below.

   (c) The number of pairs (i, j) with $i + j = n$ is given by $n - 1$ for $2 \leq n \leq 7$ and by $13-n$ for $7 \leq n \leq 12$. So, letting Z denote the sum of the values of the two rolls, we have $P(Z = 2) = 1/36$, $P(Z = 3) = 2/36$,

$P(Z=4)=3/36$, $P(Z=5)=4/36$, $P(Z=6)=5/36$, $P(Z=7)=6/36$, $P(Z=8)=5/36$, $P(Z=9)=4/36$, $P(Z=10)=3/36$, $P(Z=11)=2/36$, and $P(Z=12)=1/36$. The expected value of Z is $(2)(1/36)+(3)(2/36)+(4)(3/36)+(5)(4/36)+(6)(5/36)+(7)(6/36)+(8)(5/36)+(9)(4/36)+(10)(3/36)+(11)(2/36)+(12)(1/36)=252/36=7$. We can also do this a simpler way: since Z is the sum of the values of the first roll and second roll, linearity of expectation tells us that the expected value of Z is equal to $7/2$ (the expected value of the first roll) plus $7/2$ (the expected value of the second roll), or 7.

(d) Letting W denote the value of the first roll minus the value of the second roll, we have $P(W=5)=1/36$, $P(W=4)=2/36$, $P(W=3)=3/36$, $P(W=2)=4/36$, $P(W=1)=5/36$, $P(W=0)=6/36$, $P(W=-1)=5/36$, $P(W=-2)=4/36$, $P(W=-3)=3/36$, $P(W=-4)=2/36$, and $P(W=-5)=1/36$. The expected value of W is $(5)(1/36)+(4)(2/36)+(3)(3/36)+(2)(4/36)+(1)(5/36)+(0)(6/36)+(-1)(5/36)+(-2)(4/36)+(-3)(3/36)+(-4)(2/36)+(-5)(1/36)=0$. We can also see this using linearity of expectation, as in (c): $7/2-7/2=0$.

2. With probability $\frac{18}{38}$, the gambler wins \$1 right away. Otherwise, the gambler makes two additional bets, and winds up with total winnings of \$1, -\$1, or -\$3, with respective probabilities $\frac{20}{38}\frac{18}{38}\frac{18}{38}$, $\frac{20}{38}(\frac{18}{38}\frac{20}{38}+\frac{20}{38}\frac{18}{38})$, and $\frac{20}{38}\frac{20}{38}\frac{20}{38}$.

   (a) $P(X>0)=\frac{18}{38}+\frac{20}{38}\frac{18}{38}\frac{18}{38}=4059/6859\approx 0.5918$, which is significantly greater than $\frac{1}{2}$.

   (b) "Winning isn't everything; how *much* you win is everything!" When $X>0$ you win a dollar; but when $X<0$ you sometimes lose three dollars (not just one). So, even though $X>0$ more than half of the time, it doesn't follow that in the long run you'll make money using the strategy.

   (c) $E(X)=(1)(\frac{4059}{6859})+(-1)(\frac{1800}{6859})+(-3)(1000/6859)=-741/6859=-39/361\approx-0.108$. So in the long run, you'll lose roughly eleven cents for every time you use this strategy.

3. This one is slightly tricky, because you don't necessarily have to have tested the 2 defective units in order to know where they are. There are

ten equally likely outcomes in this sample space; we'll denote them by $DGDGG$, etc., where D means defective and G means good. Let T denote the number of tests. Then $T(DDGGG) = 2$, $T(DGDGG) = 3$, $T(DGGDG) = 4$, $T(DGGGD) = 4$, $T(GDDGG) = 3$, $T(GDGDG) = 4$, $T(GDGGD) = 4$, $T(GGDDG) = 4$, $T(GGDGD) = 4$, and $T(GGGDD) = 3$. Hence $P(T = 2) = 1/10$, $P(T = 3) = 3/10$, and $P(T = 4) = 6/10$, so $E(T) = (2)(1/10) + (3)(3/10) + (4)(6/10) = 35/10 = 3.5$.

4. First solution: Let X denote the number of defective items in the sample. $P(X = 0) = \binom{4}{0}\binom{16}{3}/\binom{20}{3}$, $P(X = 1) = \binom{4}{1}\binom{16}{2}/\binom{20}{3} = 8/19$, $P(X = 2) = \binom{4}{2}\binom{16}{1}/\binom{20}{3} = 8/95$, and $P(X = 3) = \binom{4}{3}\binom{16}{0}/\binom{20}{3} = 1/285$, so $E(X) = (1)(8/19) + (2)(8/95) + (3)(1/285) = 3/5$.

   Second solution: Imagine that we choose the items sequentially. The probability that the first item is defective is $4/20 = 1/5$. The (*unconditioned*) probability that the second item is defective is also $1/5$ (by symmetry) and likewise for the third item. Let Y_i denote the indicator random variable for the event that the ith item is defective. We have just shown that $E(Y_i) = P(Y_i = 1) = 1/5$ for $i = 1, 2, 3$. Hence, by linearity of expectation, the expected number of defective items is $E(Y_1 + Y_2 + Y_3) = E(Y_1) + E(Y_2) + E(Y_3) = 1/5 + 1/5 + 1/5 = 3/5$.

5. In the event that nobody in the group has the disease (an event with probability $(.9)^{10}$), only 1 test is required. Otherwise (i.e., with probability $1 - (.9)^{10}$), 11 tests are required. The expected number of tests is $(1)(.9)^{10} + (11)(1 - (.9)^{10}) = 11 - 10(.9)^{10}$, which is 7.51 to three significant figures. If we look at the whole block of 100 people (10 groups of 10), linearity of expectation tells us that the expected number of tests is 10 times greater, or 75.1 to three significant figures. Note that, given how the question is worded, the former answer seems better, even though the latter answer is the one that is given at the end of the book.

6. The probability of a lot being accepted is $(.9)^4$, so the probability of a lot being rejected is $1 - (.9)^4 = .3439$.

7. $E(X) = (1)(1/3) + (4)(2/3) = 3$, $E(Y) = E(X) = 3$, and $E(Z) = (2)(2/3) + (5)(1/3) = 3$. By linearity of expectation, $E(X + Y) = 3 + 3 = 6$ and $E(X + Z) = 3 + 3 = 6$.

The mean error of X is $|1-3|(1/3)+|4-3|(2/3)=4/3$. Likewise for Y. The mean error of Z is $|2-3|(2/3)+|5-3|(1/3)=4/3$.

Since X and Y are independent, we have

$$P(X+Y=2)=(1/3)(1/3)=1/9,$$

$$P(X+Y=5)=(1/3)(2/3)+(2/3)(1/3)=4/9,$$

and

$$P(X+Y=8)=(2/3)(2/3)=4/9,$$

so

$$E(|X+Y-6|)=(4)(1/9)+(1)(4/9)+(2)(4/9)=16/9.$$

Since X and Z are independent, we have

$$P(X+Z=3)=(1/3)(2/3)=2/9,$$

$$P(X+Y=6)=(1/3)(1/3)+(2/3)(2/3)=5/9,$$

and

$$P(X+Y=9)=(2/3)(1/3)=2/9,$$

so

$$E(|X+Z-6|)=(3)(2/9)+(0)(5/9)+(3)(2/9)=12/9.$$

Note that the mean error of $X+Z$ is not equal to the mean error of $X+Y$, even though X,Y,Z are all independent and all have the same expected value and the same mean error.

Moral: Knowing the expected error of two independent random variables does *not* automatically tell you the expected error of their sum! (Compare this with the fact that the standard deviation of the sum of two independent random variables *is* determined by the standard deviations of the two individual random variables.)