Math 431, Assignment #11: Solutions

(due 5/10/01)

1. Chapter 7, problem 32, via the two specified methods:

   (a) Let X_i be the indicator random variable for the event

   $$\{\text{roll } i \text{ lands on } 1\}$$

   and let Y_i be the indicator random variable for the event

   $$\{\text{roll } i \text{ lands on } 2\}.$$

   $\text{Cov}(X_i, Y_j) = E(X_iY_j) - E(X_i)E(Y_j)$, which equals $\frac{1}{36} - \frac{1}{36} = 0$ when $i \neq j$ (since the ith roll and jth roll are independent) and equals $-\frac{1}{36}$ when $i = j$ (since $X_iY_j = 0$ when $i = j$). By the bilinearity of covariance, $\text{Cov}(\sum_i X_i, \sum_j Y_j) = \sum_i \sum_j \text{Cov}(X_i, Y_j) = -\frac{n}{36}$.

   (b) Define Z_i to be the number of rolls that show an i, for i going from 1 to 6. We have $\text{Var}(Z_1 + \ldots + Z_6) = 0$ since $Z_1 + \ldots + Z_6$ is always n. Hence we have $0 = \text{Cov}(Z_1 + \ldots + Z_6, Z_1 + \ldots + Z_6) = 6\text{Cov}(Z_1, Z_1) + 30\text{Cov}(Z_1, Z_2)$ (since, when you expand $\text{Cov}(Z_1 + \ldots + Z_6, Z_1 + \ldots + Z_6)$ into 36 terms, all six of the terms $\text{Cov}(Z_i, Z_i)$ have the same value as one another, and so do all thirty of the terms $\text{Cov}(Z_i, Z_j)$ with $i \neq j$). Z_1 is a binomial random variable, so $\text{Cov}(Z_1, Z_1) = \text{Var}(Z_1) = np(1-p) = \frac{5n}{36}$. Therefore $\text{Cov}(Z_1, Z_2) = -\frac{6}{30}\frac{5n}{36} = -\frac{n}{36}$.

2. Chapter 7, problem 41:

   (a) $\text{Cov}(X_1+X_2, X_2+X_3) = \text{Cov}(X_1, X_2) + \text{Cov}(X_1, X_3) + \text{Cov}(X_2, X_2) + \text{Cov}(X_2, X_3) = 0 + 0 + 1 + 0 = 1$, $\text{Var}(X_1 + X_2) = \text{Var}(X_1) + \text{Var}(X_2) = 1 + 1 = 2$, and $\text{Var}(X_2 + X_3) = \text{Var}(X_2) + \text{Var}(X_3) = 1 + 1 = 2$, so

   $$\rho_{X_1+X_2, X_2+X_3} = \frac{1}{\sqrt{2}\sqrt{2}} = \frac{1}{2}.$$

(b) $\text{Cov}(X_1+X_2, X_3+X_4) = \text{Cov}(X_1, X_3)+\text{Cov}(X_1, X_4)+\text{Cov}(X_2, X_3)+\text{Cov}(X_2, X_4) = 0+0+0+0 = 0$ and $\text{Var}(X_3+X_4) = \text{Var}(X_3) + \text{Var}(X_4) = 1+1 = 2$, so

$$\rho_{X_1+X_2, X_3+X_4} = \frac{0}{\sqrt{2}\sqrt{2}} = 0.$$

3. Chapter 7, problem 46:

$$f_{X|Y}(x|y) = \frac{e^{-x/y}e^{-y}/y}{\int_0^\infty e^{-x/y}e^{-y}/y\,dx} = e^{-x/y}/y, \quad 0 < x < \infty.$$

Hence, given $Y = y$, X is exponential with mean y, and so $E(X^2|Y = y) = 2y^2$ (see Example 5a).

4. Chapter 7, problem 49: Let X denote the number of days until the prisoner is free, and let I denote the initial door chosen. Then

$$\begin{aligned} E(X) &= E(X|I=1)(.5) + E(X|I=2)(.3) + E(X|I=3)(.2) \\ &= (2+E(X))(.5) + (4+E(X))(.3) + (1)(.2) \end{aligned}$$

Therefore $E(X) = 12$.

5. Chapter 7, problem 59:

(a) Number the red balls and the blue balls. Let X_i equal 1 if the ith red ball is selected and let it be 0 otherwise; similarly, let Y_j equal 1 if the jth blue ball is selected and let it be 0 otherwise. We have $X = \sum_i X_i$ and $Y = \sum_j X_j$, so

$$\text{Cov}(X, Y) = \text{Cov}(\sum_i X_i, \sum_j Y_j) = \sum_i \sum_j \text{Cov}(X_i, Y_j),$$

a sum of 80 identical terms. Now, for all i, j

$$E(X_i) = E(Y_j) = 12/30$$

and

$$\begin{aligned} E(X_iY_j) &= P(\text{red ball } i \text{ and blue ball } j \text{ are selected}) \\ &= \binom{28}{10} \Big/ \binom{30}{12}. \end{aligned}$$

Thus $\text{Cov}(X, Y) = 80[\binom{28}{10}/\binom{30}{12} - (12/30)^2] = -96/145$.

(b)

$$E(XY|X) = XE(Y|X) = X(12 - X)8/20$$

where the second equality above follows since, given that x red balls have been selected, there are $12 - x$ additional balls to be selected from among the 20 non-red balls, only 8 of which are blue. Now, since X is a hypergeometric random variable it follows that $E(X) = 12(10/30) = 12(1/3) = 4$ and $E(X^2) = \text{Var}(X) + [E(X)]^2 = \frac{18}{29}12(1/3)(2/3) + 4^2 = 512/29$, so

$$\begin{aligned} E(XY) &= E(E(XY|X)) \\ &= E(X(12 - X)8/20) \\ &= \frac{8 \cdot 12}{20}E(X) - \frac{8}{20}E(X^2) \\ &= \frac{96}{5} - \frac{1024}{145} \\ &= \frac{352}{29}; \end{aligned}$$

and since $E(Y) = 12(8/30) = 16/5$, we get $\text{Cov}(X, Y) = 352/29 - (4)(16/5) = -96/145$.

6. Chapter 7, problem 60:

   (a) $E(X) = E(X|\text{type } 1)(p) + E(X|\text{type } 2)(1 - p) = p\mu_1 + (1 - p)\mu_2$.

   (b) Let I be the type (a random variable, so that μ_I and σ_I are also random variables). $E(X|I) = \mu_I$, $\text{Var}(X|I) = \sigma_I^2$, $E(\mu_I) = p\mu_1 + (1-p)\mu_2$, $E(\mu_I^2) = p\mu_1^2 + (1-p)\mu_2^2$, and $\text{Var}(X) = E(\text{Var}(X|I)) + \text{Var}(E(X|I)) = E(\sigma_I^2) + \text{Var}(\mu_I) = E(\sigma_I^2) + E((\mu_I)^2) - (E(\mu_I))^2 = p\sigma_1^2 + (1 - p)\sigma_2^2 + p\mu^2 + (1 - p)\mu_2^2 - (p\mu_1 + (1 - p)\mu_2)^2$.

7. Chapter 7, problem 61:

$$\begin{aligned} E(X^2) &= \frac{1}{3}(E(X^2|Y = 1) + E(X^2|Y = 2) + E(X^2|Y = 3)) \\ &= \frac{1}{3}(9 + E((5 + X)^2) + E((7 + X)^2)) \\ &= \frac{1}{3}(83 + 24E(X) + 2E(X^2)) \\ &= \frac{1}{3}(443 + 2E(X^2)) \text{ since } E(X) = 15. \end{aligned}$$

Hence $E(X^2) = 443$ and $\text{Var}(X) = 443 - (15)^2 = 218$.

8. Chapter 7, theoretical exercise 19:

$$\begin{aligned}
\text{Cov}(X+Y, X-Y) &= \text{Cov}(X,X) + \text{Cov}(X,-Y) + \text{Cov}(Y,X) + \text{Cov}(Y,-Y) \\
&= \text{Var}(X) - \text{Cov}(X,Y) + \text{Cov}(Y,X) - \text{Var}(Y) \\
&= \text{Var}(X) - \text{Var}(Y) = 0.
\end{aligned}$$

9. Chapter 7, theoretical exercise 22: $\text{Cov}(X,Y) = b\,\text{Cov}(X,X) = b\,\text{Var}(X)$ and $\text{Var}(Y) = b^2\ \text{Cov}(X,X) = b^2\ \text{Var}(X)$, and so

$$\rho_{X,Y} = \frac{b\ \text{Var}(X)}{\sqrt{b^2}\ \text{Var}(X)} = \frac{b}{|b|}.$$

10. Chapter 7, theoretical exercise 35:

(a) Let Y equal 1 or 0 according to whether or not $X < a$. Then

$$\begin{aligned}
E(X) &= E(X|Y=1)P(Y=1) + E(X|Y=0)P(Y=0) \\
&= E(X|X<a)P(X<a) + E(X|X \geq a)P(X \geq a).
\end{aligned}$$

(b)

$$\begin{aligned}
E(X) &= E(X|X<a)P(X<a) + E(X|X \geq a)P(X \geq a) \\
&\geq E(X|X \geq a)P(X \geq a) \\
&\geq E(a|X \geq a)P(X \geq a) \\
&= aP(X \geq a)
\end{aligned}$$

where the first inequality follows from the fact that $E(X|X < a) \geq 0$ (which in turn follows from the fact that $P(X \geq 0) = 1$).