Combinatorial Games under Auction Play

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September 14, 1997, 18:07

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Dedicated to David Ross Richman, 1956–1991

Abstract

A *Richman game* is a combinatorial game in which, rather than alternating moves, the two players bid for the privilege of making the next move. The theory of such games is a hybrid between the classical theory of games [von Neumann, Morgenstern, Aumann, . . .] and the combinatorial theory of games [Berlekamp, Conway, Guy, . . .]. We expand upon our previous work by considering games with infinitely many positions, and several variants including the *Poorman variant* in which the high bidder pays the bank (rather than the other player). The algorithmic complexity of our procedure for computing optimal moves is found to be polynomial in several important cases.

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J. Econ. Lit. Classification: C7 (game theory and bargaining theory)

AMS Subject Classification: 90D05 (2 person games)

Key Words: Combinatorial game theory, infinite directed graphs, harmonic functions.

List of symbols

- $\mathcal{L}$ — calligraphic L
- $\mathbf{B}$, $\mathbf{\scriptstyle B}$ — boldface B
- $\mathbf{N}$ — boldface N
- $\mathbf{R}$ — boldface R
- α — Greek alpha
- β — Greek beta
- γ — Greek gamma
- δ — Greek delta
- Δ — capital Greek delta
- ϵ — Greek epsilon
- π — Greek pi
- τ — Greek tau
- ω — Greek omega
- ξ — Greek xi

	Rock	Paper	Scissors
Rock	0	-1	$+1$
Paper	$+1$	0	-1
Scissors	-1	$+1$	0

Nim

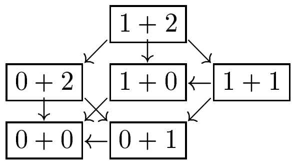


Figure 1: A typical matrix game (Rock-Paper-Scissors) and a typical combinatorial game (Nim)

1 Introduction

What's in a Game?

There are several traditions within game theory, and two in particular are largely disjoint in the published literature.

One of these traditions is now sometimes referred to as matrix game theory or classical game theory and is the subject of the famous von Neumann and Morgenstern (1944) treatise In many such games (such as Rock-Paper-Scissors, see Figure 1), two players make simultaneous moves and a payment is made from one player to the other depending on the chosen moves. Optimal strategies often involve randomness and concealment of information.

The other game theory is the combinatorial theory of the popular book *Winning Ways* (Berlekamp *et al.* 1982), with origins in the work of Sprague (1936) and Grundy (1939), and largely expanded upon by Conway (1976). In a combinatorial game (such as Nim, see Figure 1), two players move alternately and there is no hidden information. We may assume that each move consists of sliding a token from one vertex to another along an arc in a directed graph. A player who cannot move loses. If the underlying graph is finite and acyclic, then one of the players has a deterministic strategy that guarantees that player the win. If the underlying graph is infinite or contains cycles, then there is also the possibility that both players might have deterministic strategies that guarantee that they will never lose. (See Fraenkel (1994) for an exhaustive bibliography of research in combinatorial game theory.)

We do not mean to imply that these are the only two branches of the theory of games, or that they has been no interaction between them. (In particular, there

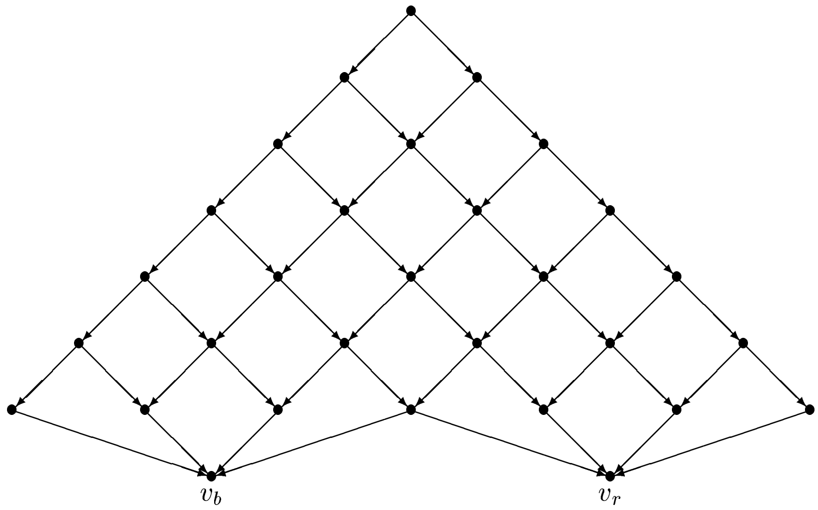


Figure 2: World Series or Quincunx Graph

is a well-developed theory of stochastic games as described in Condon (1989).) Our work should be seen as lying somewhere between these two traditions, drawing on both for inspiration.

Am I Blue?

The *spinner game* is a simple stochastic game played by two players (Mr. Blue and Ms. Red). There is an underlying combinatorial game in which a token rests on a vertex of some directed graph. There are two special vertices, denoted by v_b and v_r. (See, e.g., Figure 2.) Blue's goal is to bring the token to v_b and Red's goal is to bring the token to v_r. At each turn, a spinner divided into two equal sectors (one red and one blue) is used to determine which player has the right to make a move. Half of the time Blue is allowed to move, and half of the time Red is allowed to move. The game is a draw if neither distinguished vertex is ever reached.

It is clear that in a general position, neither player can force a win. However, the players might hope to adopt a strategy that maximizes their *probability* of winning.

If the underlying graph is infinite or contains cycles, then there is the possibility of a *draw*. Are there certain trade-offs to be made between draws and wins? For example, are there certain "risky" moves that increase the probability of a win at the expense of also increasing the probability of a loss? In other words, is the solution dependent on the relative value of a draw versus a win for each player?

If I were a Richman

Now consider the following game introduced by David Ross Richman in the mid-1980s. Each of the two players (Mr. Blue and Ms. Red) has some money. As before, Blue's goal is to bring the token to v_b on some directed graph and Red's goal is to bring the token to v_r. Instead of taking turns or randomly determining who moves next, the two players repeatedly bid for the right to make the next move. Each player secretly writes on a card a nonnegative real number no larger than the number of dollars he or she has; the two cards are then revealed simultaneously. Whoever bids higher pays the amount of the winning bid to the opponent and moves the token from the vertex it currently occupies along an arc of the directed graph to a successor vertex. Should the two bids be equal, the players alternate making moves with Ms. Red being allowed to move on the first occurrence of equal bids.

The game ends when one player moves the token to one of the distinguished vertices. The sole objective of each player is to have the game end with the token on his or her vertex. (At the end of the game, money loses all value.)

One might expect subtle psychological factors (bluffing, etc.) and mixed Nash equilibrium to be relevant to the formulation of strategies in Richman and spinner games. Surprisingly, this is not the case, for there exist optimal deterministic strategies for each player. Thus the precise manner in which the bidding is conducted becomes entirely irrelevant. Moreover, there are no tradeoffs to be made in these games, since we describe a strategy which simultaneously maximizes one's probability of winning and minimizes one's probability of losing. For these reasons, our work has only a distant connection to the substantial body of literature on the theory of auctions, as surveyed for example by Wilson (1992).

In fact, Richman and spinner games are closely related.

The probability $r(v)$ of Red winning a spinner game from a given position v, assuming optimal play from both players, is exactly the critical fraction of the total money supply that Blue needs to exceed in order to avoid a loss in a Richman game, again assuming optimal play from both players. Moreover, the probability $R(v)$ of Red avoiding a loss in a Richman game from a given position v is exactly the critical fraction $R(v)$ of the total money supply that Blue needs to exceed in order to force a win.

In the case of a finite graph, Lazarus *et al.* (1996) showed that the critical fractions $r(v)$ and $R(v)$ are equal, and we called this quantity the *Richman cost* of the vertex v. Thus, draws occur in a finite spinner game only with probability zero. Moreover, draws can only occur in the finite Richman game when the money is divided in a ratio of exactly $R(v)$ to $1 - R(v)$. In that case, optimal play results in repeated tie bids, and the fate of the Richman game is the same as the fate of the underlying combinatorial game. (The game could be a draw only if the graph has cycles.)

Stay tuned for more details

In Section 2, we review the properties of the Richman cost function of a directed graph. In particular, we relate its definition to that of a harmonic function. In Section 3, we examine a "Poorman's" variant of the Richman game, and prove some analogous results. In Section 4, we discuss various issues related to the complexity of the calculation of $R(v)$. The theorems of Lazarus *et al.* (1996) are limited to the case of finite directed graphs; in Section 5 we extend their results to the case of infinite directed graphs. Here the critical fractions $r(v)$ and $R(v)$ are not necessarily identical, so there is a non-trivial *Richman interval* in which neither player can force a win in the Richman game, and a positive probability that neither player win in the spinner game.

Several new variants of Richman/Poorman games are studied in Section 6. Surprisingly several of our results carry over to many variants. For example, the Marksman variant where one player pays in U. S. dollars and the other in deutsche marks is the Richman analogue of a spinner game played with a biased dial in which the red and blue portions of the dial do not both measure 180 degrees. On the other hand, some variants, such as Thief where both the high and the low bidder pay their bids to the bank, seem very difficult to analyze, and optimal play often dictates the use of mixed (non-deterministic) strategies.

For completeness, several results from Lazarus *et al.* (1996) are mentioned below. However, unless specifically noted, all results here are new.

2 Richman Costs

Throughout this paper, D denotes a directed graph (V, E) with a distinguished blue vertex v_b and a distinguished red vertex v_r. All other vertices are considered to be colored black. We assume that from every vertex there is either a path to v_r or a path to v_b. We assume further that every vertex has a finite number of successors. (Occasionally, we explicitly make the additional assumption that D itself has a finite number of vertices.)

These conditions guarantee that all black vertices have successors. Thus, the game can only terminate at v_b or v_r with a win for one of the two players. On the other hand, it is possible for a game to go on forever if the graph contains a cycle. If a play of the game does not terminate (that is, if the players are committed to strategies that jointly do not result in the game ever ending), then the situation is said to be a "draw".

In this section, we summarize results from Lazarus *et al.* (1996) concerning the optimal strategy in the Richman game in the case that D is finite. Since in Richman games money is paid only back and forth between the two players, the total money supply remains fixed throughout a game. It is convenient to normalize by assuming the total money supply is equal to one dollar. We assume that money is infinitely divisible.

Proofs have been omitted since they may be found in Lazarus *et al.* (1996) and in any case are similar to and simpler than the analogous proofs for the Poorman variant (Section 3).

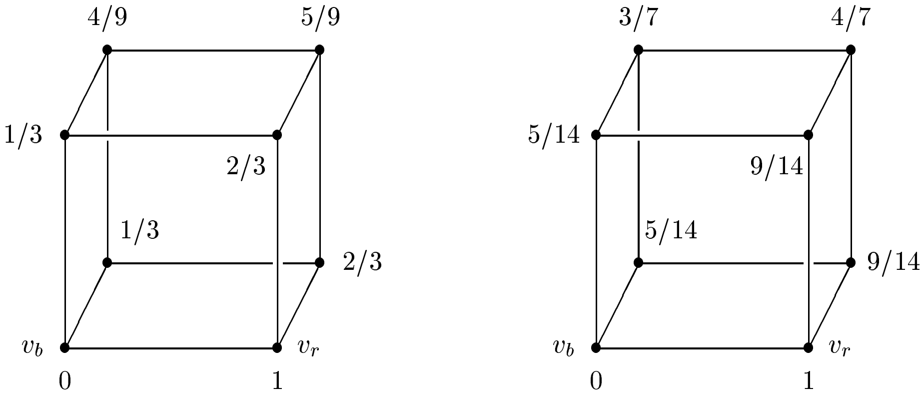
Existence

For $v \in V$, let $S(v)$ denote the set of successors of v in D, that is $S(v) = \{w \in V : (v, w) \in E\}$. Given any function $f: V \to [0, 1]$, we define

$$f^+(v) = \max_{w \in S(v)} f(w) \quad \text{and} \quad f^-(v) = \min_{u \in S(v)} f(u).$$

The key to playing the Richman game on D is to attribute costs to the vertices of D such that the cost of every vertex (except the two distinguished vertices) is the average of the lowest and highest costs of its successors. Thus, a function $R: V \to [0, 1]$ is called a *Richman cost function* if

$$R(v_b) = 0, \qquad R(v_r) = 1, \tag{1}$$



$4/9 = \frac{1}{2}(5/9 + 1/3)$ $\qquad$ $3/7 = \frac{1}{3}(5/14 + 5/14 + 4/7)$

Figure 3: Richman Cube and Harmonic Cube

and, for v black, we have

$$R(v) = \frac{R^+(v) + R^-(v)}{2}. \tag{2}$$

Note that Richman costs are a curious sort of variant on harmonic functions on Markov chains (Woess, 1994) where instead of averaging the values over *all* the successors, we average only over the two extreme values. (See Figure 3.) Every directed graph D contains a spanning subgraph D' such that the harmonic values of vertices of D' are the Richman costs of D. The subgraph D' may be found by retaining only edges of D of the form (v, v^+) and (v, v^-), where v^+ is a successor of v with $R(v^+) = R^+(v)$ (similarly for v^-). These are precisely the edges that would be chosen under optimal play, which involves Blue (respectively, Red) moving to the successor of the current vertex whose Richman cost is smallest (respectively, largest).

Theorem 1 (Lazarus ***et al.*** **(1996))** *There exists a Richman cost function $R(v)$ for any directed graph D (not necessarily finite).* $\square$

Game-theoretic interpretation

Consider the following auxiliary functions $r(v,t)$ and $R(v,t)$ with $t \in \mathbf{N}$. Let

$$r(v_b,t)=0, \qquad R(v_b,t)=0,$$

$$r(v_r,t)=1, \qquad R(v_r,t)=1,$$

for all $t \in \mathbf{N}$, and for v black, let

$$r(v,0)=0, \qquad R(v,0)=1,$$

and

$$\begin{array}{rcl} R(v,t) & = & \frac{R^+(v,t-1)+R^-(v,t-1)}{2}, \\ r(v,t) & = & \frac{r^+(v,t-1)+r^-(v,t-1)}{2} \end{array} \tag{3}$$

for $t > 0$. Their game-theoretic interpretations are given by the following result.

Theorem 2 (Lazarus *et al.* **(1996))** *Suppose Blue and Red play the Richman game on a (not necessarily finite) directed graph D with the token initially located at vertex v.*

- *If Blue's share of the total money supply exceeds $R(v) = \lim_{t\to\infty} R(v,t)$, then he has a winning strategy. Moreover, his victory requires at most t moves if his share of the money supply exceeds $R(v,t)$.*
- *If Blue's share of the total money supply is less than $r(v) = \lim_{t\to\infty} r(v,t)$, then Red has a winning strategy. Moreover, her victory requires at most t moves if Blue's share of the money supply is less than $r(v,t)$.*
- *$r(v)$ and $R(v)$ are Richman cost functions.* □

Uniqueness

We show in Section 5 that for certain *infinite* directed graphs, $r(v)$ is strictly less than $R(v)$. When Blue's share of the money supply lies in the *Richman interval* $(r(v), R(v))$, both players can prevent the other player from winning. Thus, optimal play leads to a draw.

Moreover, if D is finite, we can conclude that there is a unique Richman cost function $r(v) = R(v)$.

Theorem 3 (Lazarus *et al.* (1996)) *If the directed graph D is finite, then there is only one Richman cost function on D.* $\square$

The Richman cost $R(v)$ does indeed govern the winning strategy of a finite Richman game. In the case that D has no cycles, an optimal bid for both players is given by $\Delta = R(v) - R(v^-) = R(v^+) - R(v)$ times the total money supply, where v is the current vertex and v^- (respectively, v^+) is a successor of v with minimal (respectively, maximal) Richman cost. A player who has a winning strategy of any kind and bids Δ will still have a winning strategy one move later as long as that player is careful to move to v^- (respectively, v^+) if he (respectively, she) does win the bid. On the other hand, a player who cannot bid Δ will eventually be defeated if his opponent always plays as above.

If the graph contains cycles, then the above scheme for making bids does maintain the existence of a winning strategy, but it does not necessarily force a win! To illustrate the problem, consider the graph G in Figure 4. If Blue has

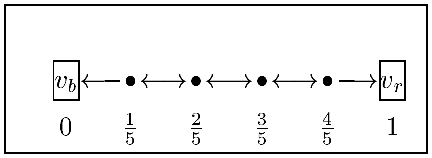


Figure 4: A directed graph with 2-cycles

$3/5$ of the money supply and the token is located at the vertex with Richman cost $1/5$, he has an obvious strategy to win in one turn. By repeatedly bidding $\Delta = 1/5$, however, Blue gives Red a chance to fend off this defeat indefinitely. She can indefinitely bid $1/5$ herself, alternately winning the move and moving the token out of danger and losing the move and collecting back her $1/5$. The token alternates between two vertices, and Blue retains his winning strategy indefinitely without ever executing it properly. To actually capitalize on his ability to win in such cases, Blue should bid $\Delta + \epsilon$ where ϵ is a nonnegative number that depends on the "distance" to the winning vertex and the difference between Blue's share of the money supply and $R(v)$. (See the proof of Theorem 11 for details.)

Corollary 4 (Lazarus *et al.* (1996)) *Suppose Blue and Red are playing the Richman game on a finite directed graph D with the token initially located at vertex v. If Blue's share of the total money supply is less than $R(v) = \lim_{t\to\infty} R(v,t)$, then Red has a winning strategy.*

Proof: Theorems 1, 2, and 3. □

Theorem 2 and Corollary 4 do not cover the critical case where Blue has exactly $R(v)$ dollars. In the critical case, with both players using optimal strategy, the Richman game reduces to the usual notion of a combinatorial game because of the way tied bids are resolved. Note, however, that in all other cases, the deterministic strategy outlined above works even if the player with the winning strategy concedes all ties and reveals his intended bid and intended move before the bidding.

Summarizing Theorem 2 and Corollary 4: in the finite case, if Blue's share of the total money supply is less (respectively, greater) than $R(v)$ in a finite Richman game, then Red (respectively, Blue) has a winning strategy.

One class of finite graphs has particularly simple Richman costs.

Corollary 5 *Let D be a finite directed graph in which every vertex that is connected to v_b is also connected to v_r and vice versa. Then for all black vertices v, $R(v) = 1/2$, i.e., whoever begins with more money wins.*

Proof: In the graph D' obtained from arbitrary D by exchanging v_b and v_r we have, for any black vertex v, $1 - r_D(v) = R_{D'}(v)$. By hypothesis, $D = D'$, and by Theorem 3, $r_D(v) = R_D(v)$, hence the corollary. □

Alternate Proof: The assignment $R(v) = 1/2$ to all black vertices is easily seen to be a Richman cost function, since any vertex with v_b as a successor has v_r as a successor by hypothesis. The corollary follows from Theorem 3. □

Now let us consider spinner games. Suppose the right to move the token is decided on each turn by the toss of a fair coin. Then induction on t shows that the probability that Red can win from the position v in at most t moves is equal to $R(v, t)$, as defined in the previous section. Taking t to infinity, we see that $R(v)$ is equal to the probability that Red can force a win against optimal play by Blue. That is to say, if both players play optimally, $R(v)$ is the chance that Red wins. The uniqueness of the Richman cost function in the finite case tells us that $1 - R(v)$ must be the chance that Blue wins. The probability of a draw is therefore zero.

If we further stipulate that the moves themselves must be random, in the sense that the player whose turn it is to move must choose uniformly at random from among the finitely many legal options, then we do not really have a game-like situation anymore; rather, we are performing a random walk on a directed graph with two absorbing vertices, and we are trying to determine the respective probabilities of absorption at these two vertices. In this case, the relevant probability function is just the harmonic function on the directed graph D or, more properly speaking, the harmonic function for the associated Markov chain (see Figure 3 and Woess, 1994).

Don't run away

Consider the directed graph depicted in Figure 5 whose Richman cost function

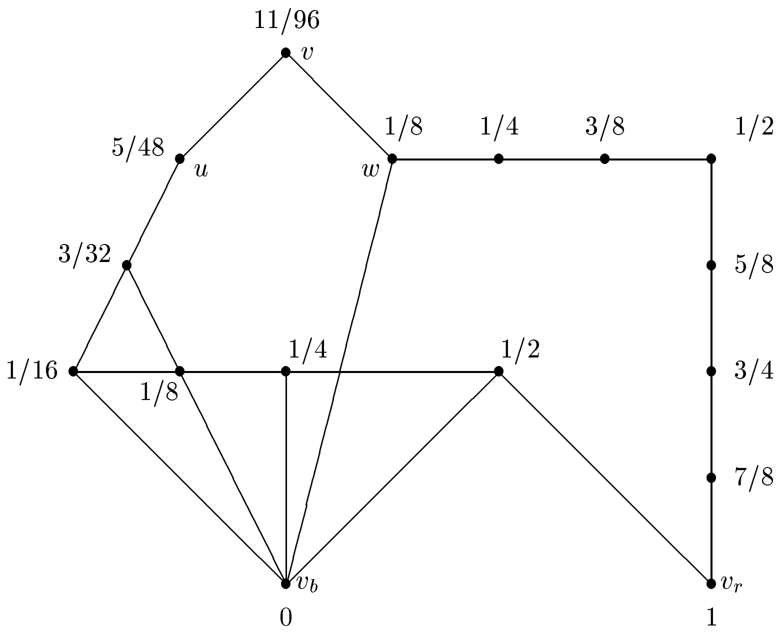


Figure 5: Optimal strategy in Richman games does not respect the usual distance function on graphs

is as indicated. If the token is located at the vertex v, then Blue can win with a fraction of the money supply greater than $11/96$. Note that his optimal move is to the vertex u, which is simultaneously farther from his goal v_b and closer to his opponent's goal v_r. His alternative is to move to the vertex w, which is closer to his goal and farther from his opponent's. This illustrates that the optimal strategy in Richman games does not respect the usual distance function on graphs.

3 Poorman Costs

When the third author first heard about Richman games, he had a slight misunderstanding concerning the rules. He thought that the higher bidder was to pay the bank instead of the lower bidder, so that the money would never be seen again.

The main lines of the above theory are still valid even in this "Poorman" variant. However, the winning strategy is governed by a different sort of cost function.

Existence

Given $0 \le x \le y \le 1$, define the *Poorman's average* of x and y to be

$$\mathrm{avg}_P(x, y) = \frac{y}{1 - x + y}.$$

Note that $\mathrm{avg}_P(x, y) \le y$ since $1 - x + y \ge 1$. Also, $x - \mathrm{avg}_P(x, y) = (y - x)(1 - x) \ge 0$, so $\mathrm{avg}_P(x, y) \ge x$.

Given a directed graph D with distinguished vertices v_b and v_r, a *Poorman cost* is a function $P\colon V(D) \to [0, 1]$ such that

$$P(v_b) = 0, \qquad P(v_r) = 1, \tag{4}$$

and

$$P(v) = \mathrm{avg}_P(P^-(v), P^+(v)) \tag{5}$$

for v black.

The situation with Poorman costs is similar to the situation with Richman costs.

Theorem 6 *There exists a Poorman cost function $P(v)$ for the (not necessarily finite) directed graph D.*

Proof: Consider the auxiliary functions $p(v,t)$ and $P(v,t)$ whose game-theoretic significance is made clearer in Theorem 7. Let

$$p(v_b,t) = 0, \qquad P(v_b,t) = 0,$$

$$p(v_r,t) = 1, \qquad P(v_r,t) = 1$$

for all $t \in \mathbf{N}$. For v black, let

$$p(v,0) = 0, \qquad P(v,0) = 1,$$

and for $t > 0$ let

$$P(v,t) = \mathrm{avg}_P(P^-(v,t-1), P^+(v,t-1)),$$

and

$$p(v,t) = \mathrm{avg}_P(p^-(v,t-1), p^+(v,t-1)).$$

A simple induction shows that $P(v,t+1) \leq P(v,t)$ for all v and all $t \geq 0$. Therefore, $P(v,t)$ is weakly decreasing and bounded below by zero as $t \to \infty$, hence convergent. It is also evident that the function $P(v) = \lim_{t\to\infty} P(v,t)$ satisfies the definition of a Poorman cost function. $\square$

Similarly, $p(v,t)$ converges to a Poorman cost function $p(v)$.

Game-theoretic interpretation

The Poorman cost function we have defined does indeed govern the winning strategy.

Theorem 7 *Suppose Blue and Red play the Poorman game on the directed graph D with the token initially located at vertex v. If Blue's share of the total money supply exceeds $P(v) = \lim_{t\to\infty} P(v,t)$, then he has a winning strategy. Moreover, his victory requires at most t moves if his share of the money supply exceeds $P(v,t)$.*

Proof: It suffices to prove the result concerning $P(v,t)$. Suppose it is true for $t-1$, and let Blue have x dollars and Red have y dollars where $x/(x+y) > P(v)$. Blue bids Δ dollars where

$$\Delta = \frac{(P(v,t) - P^-(v,t-1))x}{P(v,t)(1-P^-(v,t-1))} = \frac{(P^+(v,t-1) - P(v,t))x}{P(v,t)P^+(v,t-1)}.$$

If a bid by Blue prevails, then his share of the money supply decreases but remains at least

$$\begin{aligned}
\frac{x-\Delta}{x+y-\Delta} &= \frac{x - \frac{(P(v,t)-P^-(v,t-1))x}{P(v,t)(1-P^-(v,t-1))}}{x+y-\frac{(P(v,t)-P^-(v,t-1))x}{P(v,t)(1-P^-(v,t-1))}} \\
&> \frac{x - \frac{(P(v,t)-P^-(v,t-1))x}{P(v,t)(1-P^-(v,t-1))}}{\frac{x}{P(v,t)}-\frac{(P(v,t)-P^-(v,t-1))x}{P(v,t)(1-P^-(v,t-1))}} \\
&= P^-(v,t-1).
\end{aligned}$$

Similarly, if a bid by Red prevails, then Blue's share of the money supply increases to at least

$$\begin{aligned}
\frac{x}{x+y-\Delta} &= \frac{x}{x+y-\frac{(P^+(v,t-1)-P(v,t))x}{P(v,t)P^+(v,t-1)}} \\
&> \frac{x}{\frac{x}{P(v,t)}-\frac{(P^+(v,t-1)-P(v,t))x}{P(v,t)P^+(v,t-1)}} \\
&= P^+(v,t-1).
\end{aligned}$$

By the induction hypothesis, Blue can win in either case. □

Uniqueness

As in the case of Richman games, infinite directed graphs can have a multitude of Poorman cost functions. (See Theorem 15.) In such cases, there would be a certain amount of money needed to avoid losing, and a greater amount of money needed to force a win. If D is finite, however, there is a unique Poorman function.

Theorem 8 *There is a unique Poorman cost function $P(v)$ for a finite directed graph D. Thus, if Blue's share of the current total money supply is less (respectively, greater) than $P(v)$, then Red (respectively, Blue) has a winning strategy.*

Proof: Let P be any Poorman cost function, for example the one given by Theorem 6. Note then that

$$\mathrm{avg}_P(1-x,1-y) = \frac{1-y}{(1-(1-x)+(1-y))} = 1-\frac{x}{(1-y+x)} = 1-\mathrm{avg}_P(x,y).$$

Hence $1-P$ is a Poorman cost function for the directed graph D' obtained from D by switching the red and blue vertices.

Suppose Blue has x dollars and Red has y dollars. We show that Blue can force a win if $x/(x+y) > P(v)$ and that Red can force a win if $x/(x+y) < P(v)$. Clearly, only one function $P(v)$ can have this property. In fact, it suffices to prove that Blue can force a win if $x/(x+y) > P(v)$, since Red may imagine that she is Blue playing on the graph D'. She can force a win in D whenever $x/(x+y) < P(v)$ by imitating the strategy that Blue would use to win on D' were $x/(x+y) > 1-P(v)$, that is $x/(x+y) < P(v)$.

In our proof, we use some of the techniques used in the proof of Theorem 7, where we showed that the Poorman cost function given by $P(v) = \lim_{t\to\infty} P(v,t)$ has a certain game theoretic interpretation. There we showed that Blue can force a win whenever his fraction of the money supply exceeds $P(v)$. In this proof, we prove that *any* Poorman cost function $P(v)$ on a finite directed graph has this same property.

Define Blue's *surplus* ϵ to be

$$\epsilon = x - (P(v)y)/(1-P(v)).$$

Suppose that $x/(x+y) > P(v)$. It then follows that $x > P(v)y/(1-P(v))$, which shows that ϵ is positive.

Imagine that Blue puts ϵ dollars in a slush fund under his mattress and keeps $x' = x - \epsilon$ in his wallet. We show that the x' dollars in his wallet are enough for Blue to stave off a loss indefinitely, while the ϵ dollars in his slush fund, if strategically used, allow Blue to win the Poorman game. Below, Blue's fraction of the money supply is calculated by dividing the contents x' of his wallet by the sum $x'+y$ of his wallet and Red's money supply. (In other words, we *ignore* Blue's slush fund.)

Blue can clearly win the game if he ever is allowed to make $n = |V(D)|$ moves in succession. Define

$$\Delta(v) = \frac{(P(v)-P^-(v))x'}{P(v)(1-P^-(v))} = \frac{(P^+(v)-P(v))x'}{P(v)P^+(v)}.$$

The winning strategy we describe for Blue is to bid slightly more than $\Delta(v)$ dollars when the token is at vertex v. One can think of the $\Delta(v)$ dollars as an "ordinary expense" while the extra amount is Blue's "investment" paid out of his slush fund. Blue cleverly picks a different investment on each move with the motivation that he will make a net profit within the slush fund in every sequence of bids that terminate with a successful bid by Red. Every time Red wins an auction, Blue reassesses his net worth and discovers that he can pay a dividend to his slush fund, increasing his surplus. The percentage growth of the slush fund is bounded away from zero, and hence the slush fund increases at least geometrically with the number of times Red wins the right to move. Since there is a limit (namely, x) to how big the slush fund can grow and Blue can clearly win the game if he is ever allowed to make $n = |V(D)|$ moves in succession, it follows that Blue will eventually win.

Suppose Blue bids $\Delta(v) + \alpha$ when the token is at vertex v.

- Case 1: If Blue wins the right to move, he will pay $\Delta(v)$ from his wallet and α from the slush fund and he will move the token to a vertex w with $P(w) = P^-(v)$. Now, as in the proof of theorem 7, the amount left in Blue's wallet remains the critical fraction of the money supply, since

$$
\begin{aligned}
(x' - \Delta(v))/(x' + y - \Delta(v)) &= \frac{x' - \frac{(P(v)-P^-(v))x'}{P(v)(1-P^-(v))}}{x' + y - \frac{(P(v)-P^-(v))x'}{P(v)(1-P^-(v))}} \\
&= \frac{x' - \frac{(P(v)-P^-(v))x'}{P(v)(1-P^-(v))}}{\frac{x'}{P(v)} - \frac{(P(v)-P^-(v))x'}{P(v)(1-P^-(v))}} \\
&= P^-(v) = P(w).
\end{aligned}
$$

  Also Blue's surplus is reduced by α.

- Case 2: If Red wins the right to move, then she only has at most $y - \Delta - \alpha$ dollars left and will move to a vertex w with $P(w) \leq P^+(v)$. Even if Red pays only $\Delta(v)$ dollars to the bank, Blue's fraction of the money supply becomes

$$
\begin{aligned}
\frac{x'}{x' + y - \Delta(v)} &= \frac{x'}{x' + y - \frac{(P^+(v)-P(v))x'}{P(v)P^+(v)}} \\
&= \frac{x'}{\frac{x'}{P(v)} - \frac{(P^+(v)-P(v))x'}{P(v)P^+(v)}} \\
&= P^+(v) \geq P(w).
\end{aligned}
$$

Red actually has to pay at least α dollars more than that, so Blue can transfer a dividend of at least $\alpha P(w)/(1-P(w))$ dollars from his wallet to his slush fund and still maintain in his wallet a fraction of the money supply greater than the Poorman cost at the vertex w. This implies that we cannot have $P(w)=1$. Consequently, Red cannot win on this move.

We show that Blue, as long as he keeps winning the right to move, can pick an increasing sequence $\alpha_1, \alpha_2, \ldots, \alpha_n$ of consecutive investments with the following properties:

1. He has enough in the slush fund to pay out all of the investments, and
2. Whenever Red decides to stop this sequence of moves by outbidding him, the dividend that Blue transfers to his slush fund on that move outweighs all of the investments he has made up to that point.

Let m be the smallest nonzero value taken by the function $P(w)/(1-P(w))$. Let $r = 1 + (2/m)$ and define $\alpha_1 = 2\epsilon/m(r^n - 1)$ and $\alpha_i = \alpha_1 r^{i-1}$.

To see that Blue has enough money in his slush fund to pay all of the investments, note that

$$\begin{aligned}
\alpha_1 + \cdots + \alpha_n &= \alpha_1(1 + r + r^2 + \cdots + r^{n-1}) \\
&= \alpha_1(1 - r^n)/(1 - r) \\
&= \alpha_1(r^n - 1)/(2/m) \\
&= \epsilon.
\end{aligned}$$

Now suppose Red gets the right to move on the ith bid. Then she pays the bank, moves the token (but she cannot win on this move), and Blue transfers a dividend of at least $m\alpha_i$ dollars to his slush fund, since $m \leq P(w)/(1-P(w))$. Up to this point, Blue has already invested from his slush fund

$$\begin{aligned}
\alpha_1 + \cdots + \alpha_{i-1} &= \alpha_1(1 + r + \cdots + r^{i-2}) \\
&= \alpha_1(r^{i-1} - 1)/(2/m) \\
&\leq (m/2)\alpha_1 r^{i-1} \\
&= m\alpha_i/2,
\end{aligned}$$

so Blue's slush fund makes a net profit of at least $m\alpha_i/2 \geq m\alpha_1/2$ dollars from the affair.

Hence, Blue increases his slush fund by at least a factor of

$$\frac{\epsilon + (m\alpha_1/2)}{\epsilon} = 1 + \frac{1}{(1+(2/m))^n - 1}$$

every n moves (unless he wins in the meantime). Since his slush fund cannot increase forever (it is bounded by x), Blue eventually wins. $\square$

Here is an alternative characterization of Poorman costs that highlights the symmetry between Blue and Red: Say that a position with Poorman cost $P(v)$ has *critical ratio* $P(v) : 1 - P(v)$. If the position in $S(v)$ most favorable to Blue has critical ratio $\alpha : \beta$ with $\alpha + \beta = 1$, and the position in $S(v)$ most favorable to Red has critical ratio $\gamma : \delta$ with $\gamma + \delta = 1$; then the critical ratio for v itself is $\gamma : \beta$ (where $\gamma + \beta$ is not in general equal to 1).

In contrast to Richman games, there does not seem to be any simple interpretation of Poorman costs as probabilities in any sort of spinner game.

4 Complexity

Linear programming

For every black vertex v of the directed graph D, let v^+ and v^- denote successors of v for which $R(v^+) = R^+(v)$ and $R(v^-) = R^-(v)$. We can view equations (1) and (2) (of section 1) as a system $\mathcal{L}$ of linear equations in the variables $R(v)$. By Theorem 3, $\mathcal{L}$ must have a unique solution. Unfortunately, computing Richman costs is not as easy as solving $n = |V(D)|$ equations in n unknowns, because one doesn't know *a priori* which successors of v turn out to be v^- and v^+.

This approach does, however, give a conceptually simple (if computationally dreadful) way to calculate Richman costs. Fix $R(v_b) = 0$ and $R(v_r) = 1$. Next, for each black vertex v, choose two successors of v, call one v^- and the other v^+, and consider the equations $2R(v) = R(v^-) + R(v^+)$ together with the inequalities of the form $R(v^-) \le R(w)$ and $R(v^+) \ge R(w)$, where w ranges over all the successors of v. Since this linear system $\mathcal{L}'$ of equations and inequalities captures the notion of a Richman cost, $\mathcal{L}'$ has a solution for some choice (not necessarily unique) of v^+ and v^-, and this solution yields the unique Richman cost function of D. Since there are only finitely many choices for vertices v^- and v^+, one can simply step through these choices until the linear feasibility problem has a solution. Unfortunately, the number of such choices grows exponentially with respect to n, so such an approach is not computationally efficient.

Taking it easy

The following results show that in certain cases, it is simple to calculate Richman costs. For example, the following Proposition along with Theorem 12 show that at both extremes of "cyclicity" (namely, acyclic graphs and undirected graphs[1]) polynomial-time algorithms exist to calculate Richman cost functions.

Proposition 9 *If the directed graph D is acyclic, then its Richman cost function can be calculated in polynomial time.*

Proof: Assign Richman costs to the sinks v_r and v_b via equation (1). Since D has no cycles, there is some vertex v all of whose successors are already labeled. Label v via equation (2) and iterate this procedure. □

See for example Figures 2, 6, 7, and 12.

Proposition 10 *If all the black vertices of D have outdegree one or two, then the Richman cost function can be calculated in polynomial time.*

Proof: By the remarks in Section 2, the Richman cost function reduces to a harmonic function defined on the directed graph D. Thus, the problem reduces to the solution of a single system of $|V(D)| - 2$ linear equations. □

As an example, consider for example the graph in Figure 6 whose vertices are "scores" in the game of tennis. The successors of each vertex indicate the two possible results, depending on whether Blue or Red wins the next point.

In many examples, it has been the experience of the authors that the calculation of Richman costs may be greatly simplified by exploiting symmetry or other special structure of the underlying graph. For example, sometimes symmetry considerations allow one to infer that $R(v)$ must equal $1/2$ for certain vertices v. At other times, it is possible to prove that two vertices must have the same Richman cost, or else complementary Richman costs (costs that add to 1). Occasionally, it is possible to express the graph D in terms of simpler graphs via covering maps, composition (Courcelle, 1990), or subdivision of edges. We can thus reduce the problem of calculating the Richman cost function of D to that of calculating the Richman cost function of smaller graphs. Nonetheless, we do not know of any efficient (i.e., polynomial-time) algorithm to compute the Richman cost function in an arbitrary directed graph.

[1]Every edge of an undirected graph forms a cycle.

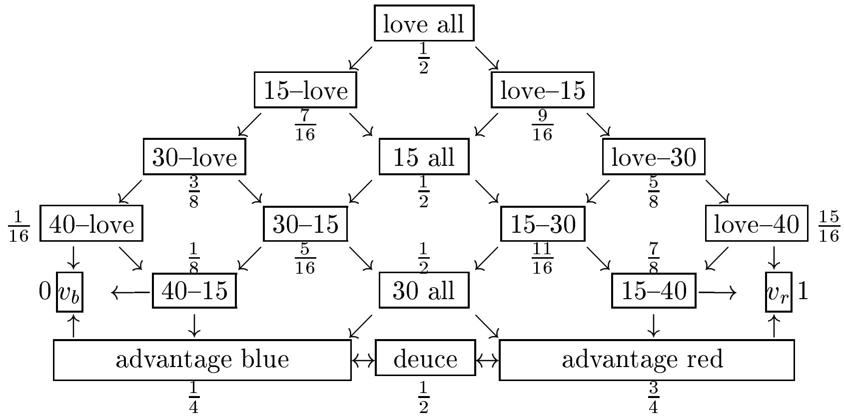


Figure 6: A Richman Game of Tennis

A curious algorithm

Another approach to the computation of Richman cost functions is to first try to approximate $R(v)$ by computing $R(v,t)$ for some appropriate choice of t and then to use that approximation to rank the vertices in order of their Richman costs. After all, once the correct ranking of vertices is known, it is an easy matter to compute the Richman costs: simply solve a system of linear equations.

We have used this approach and implemented the algorithm described in Table I to compute the Richman cost of finite directed graphs. At first glance it does not seem like an algorithm at all, since it might never halt. Surprisingly, it always does.

Theorem 11 *The algorithm in Table I calculates the Richman cost function of a finite directed graph in a finite number of steps.*

Proof: Our plan is to show that $R(v,t)$ converges to $R(v)$ "quickly", and that two non-equal Richman costs $R(v)$ and $R(w)$ cannot be "too close." Under these conditions, we can set a bound on t beyond which all strict inequalities between Richman costs are already satisfied by their approximations $R(v,t)$, and thus we have $v^+ = v_t^+$ and $v^- = v_t^-$.

1. Let $t = 0$.
2. Set $R(v, 0) = 1$ for all $v \neq v_b$, and set $R(v_b, 0) = 0$.
3. For all black vertices v, let v_t^+ (respectively, v_t^-) be a successor w of v for which $R(w, t)$ is maximal (respectively, minimal).
4. Solve the linear program $\mathcal{L}'$ under the assumption that $v^+ = v_t^+$ and $v^- = v_t^-$.
5. If there is a solution, then output it, and `stop`.
6. Otherwise, increment t.
7. Calculate $R(v, t)$ via by the recurrence

$$\begin{aligned} R(v,t) &= \tfrac{R^+(v,t-1)+R^-(v,t-1)}{2}, \\ r(v,t) &= \tfrac{r^+(v,t-1)+r^-(v,t-1)}{2} \end{aligned}$$

8. Return to step 3.

Table I: Algorithm to compute Richman cost function of a graph

Convergence is quick enough. Suppose that Blue has more than enough money to force a win in the Richman game played on the finite, directed graph $D = (V, E)$. Define Blue's *surplus* s to be the difference between his money supply and $R(v)$. Define $\Delta(v) = R(v) - R^{-}(v)$. Blue can win the game, even if D has cycles, by employing the following strategy: Until he is outbid for the first time, if the token is at vertex v on the jth move, Blue bids

$$\Delta(v) + \frac{s}{2^{n+1-j}}$$

where $n = |V|$ is the number of vertices in D.

If Red does not bid high enough to prevent n consecutive moves by Blue, then he easily wins. But if Red does outbid Blue on one of these bids, say the kth one, then Blue recalculates his surplus and finds that although his surplus decreased by

$$\frac{s}{2^n} + \frac{s}{2^{n-1}} + \frac{s}{2^{n-2}} + \cdots + \frac{s}{2^{n+2-k}} = \frac{s}{2^{n+1-k}} - \frac{s}{2^n}$$

during the first $k-1$ moves, his surplus increases on the kth move by

$$\frac{s}{2^{n+1-k}}$$

when Red earns the right to move. All told, Blue's surplus has increased by $s/2^n$ in at most n moves.

Thus Blue can either win the game or multiply his surplus by a factor of $1 + 2^{-n}$ every n turns. The surplus is bounded above by 1, so Blue can force a win in $\lceil hn \rceil$ turns whenever his surplus exceeds $(1+2^{-n})^{-h}$. Hence, by Theorem 2, $0 \le R(v,t) - R(v) \le (1+2^{-n})^{-\lfloor t/n \rfloor}$. That is, $R(v,t)$ converges "quickly" to $R(v)$.

Differences are not too small. The system $\mathcal{L}$ of n linear equations can be solved using Cramer's rule. Thus, $R(v) = \det(M_v)/\det(N)$ where M_v and N are $n \times n$ integer matrices, and the first two rows of N (corresponding to the vertices v_r and v_b) are unit vectors, and the other rows (corresponding to black vertices) consist of all zeroes except a single 2 and two -1's. Now, the determinant of a matrix is bounded by the product of the norms of its rows. Thus, $|\det(N)| \le 6^{(n-2)/2}$. Moreover, $\det(M_v)$ is an integer, so $R(v) - R(w)$ is a multiple of $1/\det(N) \ge 6^{(2-n)/2}$. That is $|R(v) - R(w)|$ is either 0 or bounded below by $6^{(2-n)/2}$.

Conclusion. The inequalities $R(v,t) \le R(w,t)$ indicate actual inequalities between Richman costs $R(v) \le R(w)$.

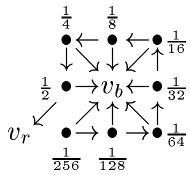


Figure 7: Distinct Richman costs can differ by as little as 2^{2-n}

After a finite number of iterations

$$t = \lceil \frac{n(n-2+\log_6(4))}{2\log_6(1+2^{-n})} + n \rceil,$$

and the algorithm terminates. Since

$$2(1+2^{-n})^{-\lfloor t/n \rfloor} < 6^{(2-n)/2},$$

any inequality $R(v,t) \leq R(w,t)$ implies $R(v) \leq R(w)$. To see this, suppose not. Then

$$\begin{array}{rcll} R(v) & \geq & R(w) + 6^{(2-n)/2} & \\ & > & R(w) + 2(1+2^{-n})^{-\lfloor t/n \rfloor}, & \text{so} \\ 2(1+2^{-n})^{-\lfloor t/n \rfloor} & < & R(v) - R(w) & \\ & = & (R(v) - R(v,t)) + (R(v,t) - R(w,t)) + (R(w,t) - R(w)) & \\ & \leq & (1+2^{-n})^{-\lfloor t/n \rfloor} + 0 + (1+2^{-n})^{-\lfloor t/n \rfloor}, & \end{array}$$

which is a contradiction.

Hence, $v^+ = v_i^+$ and $v^- = v_i^-$ and we have solved the correct system of linear equations. □

Unfortunately, our bound for the running time of this algorithm is again an exponential function of the size of the underlying graph.

Open problems

- In Figure 7, pairs of vertices exist whose Richman costs (indicated by fractions near the appropriate vertices) are not equal yet they differ by only 2^{2-n}. Can smaller differences appear? Can the bound of $6^{(2-n)/2}$ given in the proof of Theorem 11 be improved?

- One idea to calculate Poorman costs is to modify the above algorithm using $P(v,t)$ in place of $R(v,t)$. However, we know of no lower bound for the difference of distinct Poorman costs. Thus, we cannot prove that this procedure for computing Poorman costs halts in bounded time.
- It seems possible that the algorithm in Table I actually does run in polynomial time, and computational evidence suggests that it is a good algorithm in practice. One way to prove this would be to find a better strategy for winning Richman games quickly in the presence of cycles. On the other hand, perhaps the problem is NP-hard, in which case we cannot hope to find a "quick" winning strategy.
- We know that $R(v,t)$ converges to $R(v)$. However, does any sequence $R'(v,t)$ obeying equation (3) converge to $R(v)$ regardless of initial conditions? What can be said about the speed of convergence?

Undirected graphs

The remainder of this section is devoted to our polynomial-time algorithm for computing the Richman cost function of an undirected graph.

We begin by defining some useful terminology.

A *partial Richman cost function* (PRCF) on a graph G is a triple (V', E', R') where $G' = (V', E')$ is a subgraph of G containing the colored vertices v_r and v_b (not necessarily an induced subgraph) and R' is a Richman cost function of the undirected graph G'.

Our algorithm generates a sequence of partial Richman cost functions such that $R'(v)$ takes on the same value in all of the partial Richman cost functions for which it is defined. We therefore take the function R' for granted and refer to the "PRCF" G'.

The *slope* of an edge vw in any PRCF is the absolute value of $R'(v) - R'(w)$.

Given a PRCF (V', E'), a *connecting path* is a sequence

$$v_0, e_1, v_1, \ldots, e_n, v_n \ (n \geq 1)$$

of distinct vertices and edges in G such that

- each e_i is an edge joining v_{i-1} and v_i,

- v_0 and v_n are in V',
- for $1 \leq i < n$, v_i is in $V \setminus V'$, and
- for $1 \leq i \leq n$, e_i is in $E \setminus E'$.

The *length* of the connecting path is n, and the *slope* of the connecting path is $|R'(v_n) - R'(v_0)|/n$. It is possible for a connecting path to have length 1 and/or slope 0.

Theorem 12 *Suppose G is a directed graph with an edge (v, u) whenever (u, v) is an edge (that is, G is really a graph). Then its Richman cost function can be calculated in polynomial time.*

Proof: We construct an increasing sequence of PRCF's, ending with a complete Richman cost function.

We begin with the trivial PRCF (V', E') where $V' = \{r, b\}$ and $E' = \emptyset$.

The algorithm then proceeds in stages. At each stage, we find an (undirected) connecting path $v_0, \ldots, v_n$ with the largest possible slope s. Without loss of generality, we label the vertices of the path so that $R'(v_0) \geq R'(v_n)$. We now extend R' to the path by putting

$$R'(v_i) = R'(v_0) + si.$$

We write $V'' = V' \cup \{v_1, \ldots, v_{n-1}\}$ and $E'' = E' \cup \{e_1, \ldots, e_n\}$. (The edges here are taken to be undirected edges. That is, for every directed edge added to E', the reversal of that edge is also added to E'.) We need to show that (V'', E'') is a PRCF. We can then put the undirected graph (V'', E'') in place of the undirected graph (V', E') until there are no more connecting paths.

Claim: (V'', E'') is a PRCF.

It is easy to see that the Richman rule (2) is satisfied at the new vertices $v_1, \ldots, v_{n-1}$. Indeed, each such vertex has only two neighbors in (V'', E''), and its value is midway between the values of those two neighbors.

The rule (2) is clearly satisfied for vertices in V' other than v_0 and v_n since their immediate neighborhood is unchanged.

To see that (2) is also satisfied at v_0 and v_n (if they are not themselves the colored vertices v_r or v_b), we show that the slope of our new connecting path cannot exceed the slope of any edge in E'. It follows that the max and min values in (2) are not changed by the addition of new neighbors to v_0 and v_n.

Thus, we need prove only that s does not exceed the slope of any edge in E'. Each of those edges entered the PRCF by being in an earlier connecting path, and it inherited its slope from that path. So, we are really proving that the sequence of slopes of connecting paths occurring in the algorithm is non-increasing.

We show this in the context of our present notation by showing that the act of going from G' to G'' does not introduce any new connecting paths with slopes greater than s. Indeed suppose that

$$w_0, f_0, w_1, f_1, w_2, \ldots, f_m, w_m$$

is a connecting path of (V'', E'') with slope $t > s$. We assume without loss of generality that $R'(w_0) < R'(w_m)$. We divide into cases, according to the location of w_0 and w_m.

- If w_0 and w_m are both in V', then the new connecting path is actually a connecting path of G' with larger slope than the original connecting path. That contradicts the construction of G''.
- If w_0 and w_m are both in $V'' \setminus V'$ say $w_0 = v_i$ and $w_m = v_j$ for some i, j with $i < j$, then the condition $t > s$ implies that $m < j - i$. Now the path through the vertices

  $$v_0, v_1, \ldots, v_i = w_0, w_1, \ldots, w_m = v_j, \ldots, v_n$$

  is a shorter and hence steeper connecting path of G' than the one we actually found. This also contradicts the construction of G''.
- If w_0 is in V' and w_m is in $V'' \setminus V'$, say $w_m = v_j$. (The case with only w_m in V' is similar.) In this case, we consider the path through the vertices

  $$w_0, \ldots, w_m = v_j, \ldots, v_n.$$

  The slope of this path is a weighted average of t and s, hence larger than s, but it is a connecting path G' so we have once again a contradiction.

This establishes that no paths with slopes larger than s are introduced, which guarantees that the sequence of slopes in the algorithm is non-increasing as required.

Completing the algorithm.

If there are no connecting paths for G', then we are almost done. Each unlabeled vertex v is connected via edges not in E' to exactly one vertex w of G'. (At least one vertex since there must be a path in G to v_r or v_b. At most one vertex since, were there two, there would be a connecting path between the two vertices.)

We need only set $R'(v)$ to $R'(w)$. This completes the Richman labeling.

To see that this is a polynomial-time algorithm, note that each stage adds at least one edge to the PRCF and that the main work of each stage can be accomplished by one shortest-path search for each pair of labeled vertices. $\square$

The vertices in G' before the final step of the algorithm constitute the "interesting" part of the graph G. On any vertex outside of G' play must pass through a "choke point" w on the way to either v_r or v_b. Thus, neither player will see any interest in bidding until the vertex w has been reached.

Corollary 13 *Suppose G is a directed graph with an edge (v,u) whenever (u,v) is an edge (that is, G is really a graph). Suppose that w is a vertex on some minimal length path from v_b to v_r. Then $R(w) = \mathrm{dist}(v_b,w)/\mathrm{dist}(v_b,v_r)$ where $\mathrm{dist}(u,v)$ is the length of the shortest path from u to v.* $\square$

5 Finite Paths and Infinite Graphs

Tug of War

The simplest sort of finite graph on which one might want to play a Richman or Poorman game would be a path. (See for example Figure 8.) Clearly Blue

$$\begin{array}{cccccccc}
P(v) & 0 & & \frac{2}{3+\sqrt{5}} & & \frac{2}{1+\sqrt{5}} & & 1 \\
 & \boxed{v_b} & \longleftarrow & \bullet & \longleftrightarrow & \bullet & \longrightarrow & \boxed{v_r} \\
R(v) & 0 & & \frac{1}{3} & & \frac{2}{3} & & 1
\end{array}$$

Figure 8: Irrational Poorman Costs, and Rational Richman Costs

will pull the token to the left whenever he gets a chance, and Red will push the token to the right if given the opportunity. The game amounts then to a sophisticated sort of "Tug of War" played with a token instead of rope, and with money taking the place of muscle.

Corollary 13 allows us to calculate the Richman costs of all vertices. The costs increase in arithmetic progression from $R(v_b) = 0$ on the left to $R(v_r) = 1$ on the right.

The calculation of Poorman costs in tug-of-war is due to Noam Elkies.

Proposition 14 (Elkies) *Let G be path of length $n \geq 1$ with vertices labeled $v_b = 0,\ 1,\ 2,\ \ldots,\ n-1,\ n = v_r$. Then the Poorman cost is given by*

$$P(k) = \frac{\omega^{k+1} - \omega}{(\omega + 1)(\omega^{k+1} - 1)} \tag{6}$$

where ω is a primitive $(n+2)$nd root of unity.

Proof: Let $Q(k)$ denote the right hand side of equation (6).

$$\begin{aligned}
Q(0) &= \frac{\omega - \omega}{(\omega + 1)(\omega - 1)} \\
&= 0. \\
Q(n) &= \frac{\omega^{n+1} - \omega}{(\omega + 1)(\omega^{n+1} - 1)} \\
&= \frac{\omega^{n+1} - \omega}{\omega^{n+2} + \omega^{n+1} - \omega - 1} \\
&= \frac{\omega^{n+1} - \omega}{\omega^{n+1} - \omega} \\
&= 1.
\end{aligned}$$

Thus, equation (4) holds. Also,

$$\begin{aligned}
&\operatorname{avg}_P(Q(k-1), Q(k+1)) \\
&= \frac{Q(k+1)}{1 - Q(k-1) + Q(k+1)} \\
&= \frac{\dfrac{\omega^{k+2} - \omega}{(\omega + 1)(\omega^{k+2} - 1)}}{1 - \dfrac{\omega^{k} - \omega}{(\omega + 1)(\omega^{k} - 1)} + \dfrac{\omega^{k+2} - \omega}{(\omega + 1)(\omega^{k+2} - 1)}} \\
&= \frac{(\omega^{k+2} - \omega)(\omega^{k} - 1)}{(\omega + 1)(\omega^{k} - 1)(\omega^{k+2} - 1) - (\omega^{k} - \omega)(\omega^{k+2} - 1) + (\omega^{k+2} - \omega)(\omega^{k} - 1)}
\end{aligned}$$

$p(v)$	0	0	0	0	0	0	0	0	0	$\ldots$
$P(v)$	$\frac{0}{2}$	$\frac{1}{4}$	$\frac{2}{6}$	$\frac{3}{8}$	$\frac{4}{10}$	$\frac{5}{12}$	$\frac{6}{14}$	$\frac{7}{16}$	$\frac{8}{18}$	$\ldots$
	$\boxed{v_b = 0} \longleftarrow$	$\boxed{1} \longleftrightarrow$	$\boxed{2} \longleftrightarrow$	$\boxed{3} \longleftrightarrow$	$\boxed{4} \longleftrightarrow$	$\boxed{5} \longleftrightarrow$	$\boxed{6} \longleftrightarrow$	$\boxed{7} \longleftrightarrow$	$\boxed{8} \longleftrightarrow$	$\ldots$
$r(v)$	0	0	0	0	0	0	0	0	0	$\ldots$
$R(v)$	0	0	0	0	0	0	0	0	0	$\ldots$

Figure 9: The yellow brick road

$$
\begin{aligned}
&= \frac{(\omega^{k+2} - \omega)(\omega^k - 1)}{\omega^{2k+3} + \omega^{2k+2} - 2\omega^{k+2} - 2\omega^{k+1} + \omega + 1} \\
&= \frac{(\omega^{k+2} - \omega)(\omega^k - 1)}{(\omega + 1)(\omega^{2k+2} - 2\omega^{k+1} + 1)} \\
&= \frac{\omega(\omega^{k+1} - 1)(\omega^k - 1)}{(\omega + 1)(\omega^{k+1} - 1)^2} \\
&= \frac{\omega(\omega^k - 1)}{(\omega + 1)(\omega^{k+1} - 1)} \\
&= Q(k).
\end{aligned}
$$

Thus, equation (5) holds. □

Are you seeing red?

Now consider the semi-infinite path indicated by Figure 9. Clearly, the game heavily favors Blue since there is no way for Red to win. Nonetheless, Red can hope to draw.

Theorem 15 *On the semi-infinite path indicated by Figure 9, there is a unique Richman cost function, namely $R(v) = 0$ for all v. On the other hand, there are many Poorman cost functions, and at the vertex k there is a non-trivial Poorman interval* $(0, k/(2k+2))$.

Proof: Since Red can win neither game, $r(k) = p(k) = 0$ for all k.

Consider the Richman game, and suppose Blue's fraction of the total money supply is $\epsilon > 0$ and the token is on the vertex k. Let Blue play *as if* the vertex

$n = \lceil \frac{k}{\epsilon} \rceil$ were colored red. Since Blue has more than k/n of the total money supply, he can force a win on a finite path of length n and thus *a fortiori* on the infinite path.

Now consider the Poorman game, and suppose Blue has a winning strategy that guarantees a win in at most n moves for some number n. This strategy is then also a winning strategy on the finite path of length n. However, applying l'Hôpital's rule to the right hand side of equation (6), we get

$$\lim_{x\to\infty} \frac{x^{k+1}-x}{(x+1)(x^{k+1}-1)} = \frac{k}{2k+2},$$

so $P(k) > k/(2k+2)$ on the path of length n. Thus, Blue's share of the money supply must have been more than $k/(2k+2)$.

Conversely, suppose that Blue's share of the money supply is $x > k/(2k+2)$. Then $x > P(k)$ on some finite path. Since Blue can force a win on this path, he can do so on the infinite path. □

Thus, on the semi-infinite path, Blue can force a win from any vertex in the Richman game given any positive amount of money, but cannot force a win from vertex k in the Poorman game without at least $k/(2k+2)$ of the money supply.

Although the non-trivial Poorman interval may seem surprising here, there is nothing special about this particular example. One sees the same thing in any situation where the token is located at a vertex which is on an infinite path $v_0 = v_b, v_1, v_2, \ldots$ starting at the blue vertex and leading away ($\mathrm{dist}(v_i, v_0) = i$). In that situation, Red can adapt her strategy for indefinite survival on the graph of Figure 9 in an obvious way. (Of course, there may be even better strategies available to Red if the red vertex is favorably situated.)

The example given in Figure 10 shows that there are *directed* graphs with non-trivial *Richman* intervals. From $v = 0$, there is obviously a unique Richman cost $R(v) = 1/2$, since either player can immediately win with the majority of the total money supply. However, from $v = 1$ and any point "farther out" there is a non-trivial Richman interval $(1 - R(v), R(v))$. Suppose the total money supply is one dollar. From the first proof of Corollary 5, Red can force a win with more than $R(v)$ dollars. With under $1 - R(v)$ dollars, Red cannot avoid a Blue win. However, with some amount of money strictly between $1 - R(v)$ and $R(v)$ dollars, Red can force a draw but not a win.

In this particular case, $R(v)$ can be calculated thanks to a formula of Kemperman (1961) for left-continuous random walks.

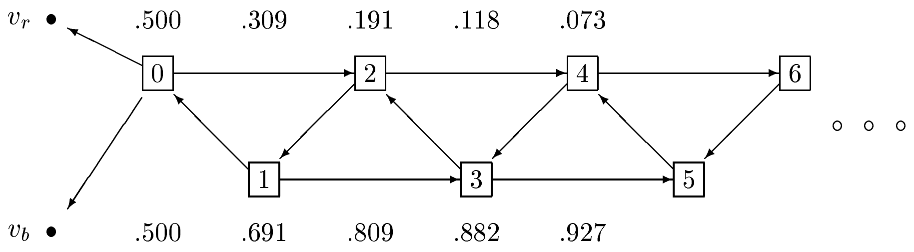


Figure 10: Two steps forward, one step back

Theorem 16 *Consider the directed graph D depicted in Figure 10 whose black vertices are labeled $0,\ 1,\ 2,\ 3,\ \ldots$ where the successors of k ($k > 0$) are $k-1$ and $k+2$, and the successors of 0 are the colored vertices v_b and v_r. Then for $k > 0$, D has a nontrivial Richman interval extending from*

$$r(k) = \sum_{i=0}^{\infty} 2^{-k-3i} \binom{3i+k}{i} \frac{k}{3i+k}, \tag{7}$$

to $R(k) = 1 - r(k)$.

Proof: Using the equivalence between the spinner game and Richman's game, and taking note of the obvious optimal strategies for Blue and Red, we can see that $r(k)$ is the probability of eventually returning home (from k to -1) on a random walk of step lengths -1 and $+2$. Kemperman (1961) shows that the probability of returning home (from k to -1) for the first time on step n is equal to k/n times the probability of being at home on step n (starting from k) assuming that the walk continues forever on $\mathbf{Z}$ (not on $\mathbf{N}$).

The only way to return home in n steps is to take i steps right and $k+2i$ steps left, where $n = k+3i$.

Summing over all values of i, we get the desired result. As in the proof of Corollary 5, the symmetry in v_b and v_r implies that $R = 1 - r$. □

The difference between $R(k)$ and $r(k)$ indicates how much easier it is to move the token to the right (and draw) than to move the token to the left (and win). Note that equation (7) is just

$$r(k) = 2^{-k}\, {}_3F_2 \left(\begin{matrix} \frac{k}{3}, \frac{k+1}{3}, \frac{2k+1}{3} \\ \frac{k+1}{2}, \frac{k+2}{2} \end{matrix} ; \frac{27}{32} \right)$$

where ${}_3F_2$ is Barnes's extended hypergeometric function. This allowed us to use Maple's facility for hypergeometric functions to compute the values of R and r in Figure 10. Also note that looking at how many terms are needed to pass Red's money supply in the sum in Theorem 16 indicates how quickly Blue can win.

Getting real

On a finite directed graph, Richman costs are always rational, since the costs can be computed by solving a system of linear equations of the form $R(v) = (R(v^+) + R(v^-))/2$ where v ranges over $V(D)$.

Nevertheless, Richman costs on an infinite directed graph are arbitrary reals between 0 and 1. In Figure 11, we illustrate how the real number $\pi - 3$ can be represented by an infinite directed graph. (The vertices labeled v_b and v_r should be identified.)

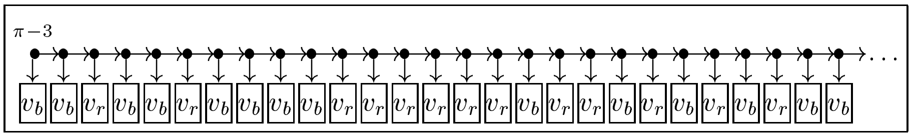


Figure 11: The Richman representation of the number $\pi - 3 \approx .1415926_{decimal}$ $\approx .001001000011111101101010100_{binary}$.

The multiplication of real numbers has a satisfying graph-theoretic interpretation. Let x and y be real numbers represented by Richman games D_x and D_y. Their product is the Richman cost of the root of the graph obtained by substituting (in the sense of graph grammars, see Courcelle, 1990) the directed graph D_y for v_r in the directed graph D_x.

This substitution is also useful on finite graphs. For example, the substitution of the directed graph in Figure 6 (corresponding to a *game* of Richman tennis) in the nodes of an appropriate graph gives a larger graph corresponding to a *set* of Richman tennis or a *match* of Richman tennis.

As in the example of Figure 8, Poorman costs may be irrational even on a finite directed graph. However, Poorman costs on a finite directed graph are always algebraic since they constitute an isolated (actually the unique) real solution of a system of polynomial equations with rational coefficients. (See Lemma 3.4, proved by M. Matignon, in Berge, 1995.)

Undirected graphs

The examples in Figures 9 and 10 show that there are undirected graphs with non-trivial Poorman intervals and that there are directed graphs with non-trivial *Richman* intervals. However, there are no locally finite undirected graphs with non-trivial Richman intervals.

Theorem 17 *Suppose G is a locally finite directed graph with an edge (u, v) whenever (v, u) is an edge (that is, G is really an undirected graph). Then G has a unique Richman cost function.*

Proof: We first show how the algorithm given in the proof of Theorem 12 can be extended to infinite graphs. It then suffices to show that any Richman cost function must be equal to the Richman cost function so constructed.

Algorithm. We must show that steepest connecting paths actually exist, and that union of the connecting paths of nonzero slope form a subgraph $G' = (V', E')$ such that for all $v \notin V'$, the vertices $w \in V$ connected to v by edges not in E' all have the same partial Richman cost.

Suppose that G' is a subgraph that does not meet this condition. Then there is some vertex v not in V' connected to two vertices u, w of V' with $R'(u) \neq R'(w)$. Thus, there is a connecting path p from u to w of nonzero slope s. To find a steepest such path, we need only consider vertices v' within distance $1/s$ of both v_r and v_b. There are only finitely many such vertices since G is locally finite. Hence, the steepest path $\hat{p}$ is well defined (and gets added to G').

If p and $\hat{p}$ share no edges, then the next steepest path again lies in the set S of vertices within distance $1/s$ from v_r and v_b. It is this path that is added next to G'. Since the set S contains a finite number of edges, an edge of p must eventually be added to G'. Hence, every connecting path will eventually have one of its edges added. Any connecting paths remaining from the edges of the original connecting path must be shorter than the complete path. Since connecting paths are of finite length, they can be shortened only a finite number of times.

When no more connecting paths of nonzero slope exist, the algorithm may be completed as in Theorem 12.

Uniqueness. Let R be the Richman cost function defined by the algorithm above, and let R' be another Richman cost function. Let v be the first vertex encountered by the algorithm that is assigned a cost $R(v) \neq R'(v)$. Either v was added as part of a connecting path, or v was added during the final step of the algorithm.

Suppose v was added during the final step of the algorithm. Then there is a subgraph $G' = (V', E')$ such that for all $w \in V'$ to which v is connected by a path of edges not in E' we have $R(v) = R(w) = R'(w)$. However, by pretending to play the game "tug of war" analyzed above, either player can bring the token to the vertex w at an arbitrarily small cost to himself. It then follows that $R'(v) = R(w)$.

Otherwise, v was added to a subgraph G' as part of a connecting path of slope s: $v_0, \ldots, v, \ldots, v_n$. Without loss of generality, v is the first v_i on the path such that $R(v_i) \neq R'(v_i)$, since all of these vertices were assigned a cost "simultaneously."

Consider the Richman costs R'. Suppose that some edge w_0w_1 not in E' had slope $\Delta > s$ according to R'. Without loss of generality, w_0w_1 is the edge of greatest ascent from w_0. Then extend this edge to a *path of greatest ascent* $w_0w_1w_2\ldots$ by choosing an edge w_1w_2 of greatest ascent from w_1, and an edge w_2w_3 of greatest ascent from w_2 and so on. Such a path must terminate with v_r which is by definition in V'. So let w_m be the first vertex of the path in V'. Similarly, construct a path $v = w_0, w_{-1}, w_{-2}, \ldots, w_{-k}$ of steepest descent where w_{-k} is the first vertex on the path in V'. Each edge in this path has slope at least Δ by equation (2), hence the path has slope at least $\Delta > s$. However, by definition, there is no connecting path of slope greater than s.

Thus, there is no edge of slope greater than s outside of E'. Hence, v_0 and v_n are separated by a path of n edges of slope at most s. Now, $R(v_n) - R(v_0) = ns$ so the slope of each of the n edge must be exactly s. Thus, $R'(v_i) = R'(v_0) + is$. $\square$

Note that Theorem 17 is not a simple consequence of the fact that Richman cost functions of *finite* undirected graphs are easily computed. In fact, Richman cost functions of finite acyclic directed graphs are also easily computed, yet infinite acyclic directed graphs can have non-trivial Richman intervals.

Open problem: Calculate the Richman (or Poorman) cost function of more interesting infinite graphs. In particular, study the infinite lattices (square, triangular, hexagonal, cubic, ...). Can the Richman cost function of certain finite subgraphs be used to approximate the Richman cost function of the entire graph?

6 Other Variants

Taxman

The Richman and Poorman games studied above are both special cases of the *Taxman game*. In the Taxman game, players bid as above for the right to move. The winner pays his bid (say Δ dollars). Of this amount, the government takes $\tau\Delta$ dollars, and his opponent takes $(1-\tau)\Delta$ dollars where τ is a constant known as the *tax rate*.

The Richman game is the tax-free case $\tau = 0$, and the Poorman game is the confiscatory case $\tau = 1$.

Such a game is ruled by a *Taxman cost* function defined by $T(v_b) = 0$, $T(v_r) = 1$, and $T(v) = \mathrm{avg}_T(T^-(v), T^+(v))$ where the *Taxman's average* is given by

$$\mathrm{avg}_T(x, y) = \frac{x + y - x\tau}{2 - (1 + x - y)\tau}$$

for v black and $\tau \leq 2$ (we explain below what is meant by putting $\tau > 1$ or $\tau < 0$). In the acyclic case, optimal bids for both players are given by

$$\Delta = \frac{T^+(v) - T^-(v)}{2 - (1 - T^-(v) + T^+(v))\tau}$$

(expressed in units equal to the total money supply).

If $\tau > 1$, then the game is not well defined unless we indicate what happens if the low bidder does not have enough money to cover his or her share of the opponent's bid, $(\tau - 1)\Delta$ dollars. In such a case, we say that the lower bidder *defaults*, and the higher bidder is allowed to make all remaining moves for free. The game is still (somewhat) playable until $\tau = 2$. At that point (*maximal taxation*), *both* players must pay the high bid, and the high bidder makes a

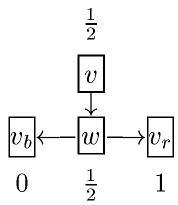


Figure 12: There is such a thing as a free move: optimal bid at v is zero.

move. Clearly for $\tau \geq 2$, it is in both players' interests to always bet all of their money. In fact for $\tau = 2$, $T(v) = 1/2$ at all black vertices indicating that the player with more money wins. That is, neither player has an advantage.

On the other hand, $\tau < 0$ does lead to a possible game, better called *Doleman*, since the low bidder obtains the entire bid of his opponent (as in the Poorman game) plus a *government subsidy* of $(-\tau)\Delta$ dollars.

Beggarman

The rules of the Richman game and its variants above explicitly forbid negative bids in which one actually begs the opponent to move next (compare Berlekamp, 1996). Even in a *Beggarman* variant in which such bids were allowed, there would never be a reason to make a negative bid: since all successor vertices are available to both players, it cannot be preferable to have the opponent move next. That is to say, there is no reason to part with money for the chance that your opponent may carry out through negligence a move that you yourself could perform through astuteness.

Although negative bids are never be needed by optimal Beggars, a bid of zero may be optimal in certain positions, such as v in Figure 12.

One can imagine yet another variant in which the edges of the directed graph are colored blue and red, with the players restricted to moves where the token is slid along an edge of their own color. In this case, a rule to allow negative bids would certainly influence the outcome of the game. For example, if the token rested on a vertex from which all red edges lead to v_b and all blue edges lead to v_r, the players would give anything to force the "privilege" of the next move on their opponent. Some rule would have to be imposed to forbid very negative bids.

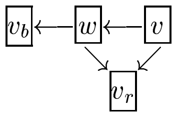


Figure 13: A game without deterministic strategies.

Thief

In *Thief,* the players bid secretly as usual, then reveal their bids. *Both* players pay their own bid to the bank. The high bidder moves.

Unlike any of the games studied above, Thief may fail to have deterministic optimal strategies.

Consider the directed graph D depicted in Figure 13 where the token is located on vertex v. Suppose the total money supply is one dollar. Clearly, Red can force a win with over 50 cents, and Blue can force a win with over $66\frac{2}{3}$ cents, outbidding Red at both v and w. However, what happens with intermediate holdings?

First suppose that Blue has $\mathbf{B} < 66\frac{2}{3}$ cents. If he plays the pure strategy $x < 33\frac{1}{3}$, he will lose to the Red strategy of $33\frac{1}{3}$ cents every time. On the other hand, a bid of $x \geq 33\frac{1}{3}$ loses to the Red strategy of betting 0 every time. Hence Blue's pure strategies guarantee nothing. On the other hand, every Red pure strategy $y \in [0, 1 - \mathbf{B}]$ loses to Blue's betting $y + 1$ cents. Hence optimal play in this game involves mixed strategies.

If $\mathbf{B} = 66\frac{2}{3}$, the Red nondeterministic strategy of playing 0 and $33\frac{1}{3}$ each with probability 1/2 wins half the time against any Blue strategy; no deterministic strategy guarantees winning at all. Blue can guarantee winning half the time with a deterministic strategy of betting $33\frac{1}{3}$, so the (classical game-theoretic) value is 1/2.

We demonstrate the solution for the example $\mathbf{B} = 60$ in Table II. It may be seen from the first chart that if Blue plays the strategies 20 cents and 40 cents in the ratio 1 : 2 his expectation is 1/3. From the second chart we see that Red expects to win 2/3 of the time with the mixed strategy of 0, 20, and 40 cents played each with probability 1/3.

A classical matrix-type game in which the players' strategies are taken from the interval $[0, 1]$ is known as a "game on the unit square" (Owen 1991, Chapter IV). Since we assume money is infinitely divisible, the bet at each vertex

Blue \ Red	$[0,20)$	20	$(20,40)$	40
20	1	$1/2$	0	0
40	0	$1/2$	1	$1/2$

Blue \ Red	0	20	40
$[0,20)$	1	0	0
20	$1/2$	$1/2$	0
$(20,40)$	0	1	0
40	0	$1/2$	0
$(40,60]$	0	0	1

Table II: Certain strategies with their expected payoffs for Blue in Figure 13

may be seen as a game on the unit square as long as the payoff function may be calculated. We may do that with Figure 13, because the optimal moves and the value of the game at w are obvious. In general, such games may not have optimal strategies (even nondeterministic) or values. We do not know if arbitrary Thief games must have optimal strategies and values.

Tradesman

In the Thief game above, as in the Poorman game, all money goes to the bank. Instead we could play a *Tradesman game* where both bidders pay their bid to the other player. Thus, this game resembles both the Thief game (in that both players pay), and the Richman game (in that no money leaves the system). The bottom line is that the high bidder loses the difference of the two bids to the low bidder.

Consider the example in Figure 13. Using reasoning similar to that outlined above, we can see that neither player has a deterministic winning strategy under these rules.

Marksman

As usual in this game, both players write their bids secretly on a card, and the cards are then revealed simultaneously. Suppose Herr Blau writes the number x and Ms. Red writes the number y.

If $x > y$, then Blau is deemed to be the high bidder, and he pays Red x deutsche marks. If $x < y$, then Red is deemed to be the high bidder, and she pays Blau y U.S. dollars.

Suppose that a bank is available that freely trades 1 U.S. dollar for ξ deutsche marks where ξ is the *exchange rate.*

Such a game is ruled by a *Marksman cost* function defined by $M(v_b) = 0$, $M(v_r) = 1$, and $M(v) = \mathrm{avg}_M(M^+(v), M^-(v))$ where the *Marksman's average* is given by

$$\mathrm{avg}_M(x, y) = \frac{x + \xi y}{1 + \xi} \tag{8}$$

for v black. The Marksman cost $M(v)$ indicates the critical fraction of the total money supply that Blau needs in order to force a win. The fraction of the total money supply is calculated by converting all money into the same currency (either U.S. dollars or deutsche marks). Bidding a fraction $\Delta = \frac{M^+(v) - M^-(v)}{\xi + 1}$ of the total money supply guarantees a win for Blau on an acyclic graph, if he has a winning strategy at all.

Now, consider a spinner game where the red and blue portions of the dial do not both measure 180 degrees. Each turn Blau has 1 chance in $1 + \xi$ of getting the right to move, and Red has ξ chances in $1 + \xi$ of getting the right to move. The probability of winning for the vertex v is thus the Marksman's average of the lowest and highest probability of winning among the vertices following v. The Marksman game is thus clearly the Richman analogue of a game played with a biased spinner.

Dullman

This is actually a class of several variants.

The low bidder pays his bid (or else his opponent's bid) to the bank (or else to the high bidder), and the high bidder gets the right to move for free.

The rich get richer and the poor get poorer. There is no incentive of any kind to bid anything less than all of one's resources. Thus, the critical *Dullman's cost* at every black vertex from which there are paths to *both* v_b and v_r is $1/2$. Dull game.

Pseudo-variants

There are also some ways of playing a Richman game on a finite directed graph that look like genuine variants but aren't.

First, one could prevent each player from knowing how much money his or her opponent has. However, as was shown in our earlier article (Lazarus *et al.* 1996), this does not affect the costs of positions, though it does restrict the kind of strategy that a player might have to use in order to cash in on a winning situation when the directed graph D has loops.

Second, the bidding procedure could be modified in any of a number of ways, so that for instance one player bids first, then the other player must either top that bid or drop out of the bidding, and so on until one of the players drops out, in which case the other player wins the bid. In this case, too, the governing costs of positions are the Richman costs, and an ideal bid is the one given by Richman's original formula. The only situation in which the details of the bidding protocol "matter" is the critical case; otherwise, the favored player can win simply by initially bidding as in an ordinary Richman game and refusing to bid any higher.

Third, one could decree that the player who makes the higher bid pays the amount of the *lower* bid to his or her adversary. It might initially seem surprising that the theory that applies under this convention should be the same as the theory for Richman's convention. However, one can readily check that this variation only increases the advantage of the favored player. For, if this player has just made the perfect bid (of size B, say), then the player will pay less money if he or she wins the bid and the player's opponent must still pay B in order to win the bid.

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