Laurent polynomials,
dynamics,
and combinatorics

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presented at the Fellowship
of the Ring seminar
Brandeis University
November 30, 2002

(last modified November 26, 2002)

I. Laurent polynomial dynamics

Define $s_1 = x$, $s_2 = y$, and

$$s_n = (s_{n-1} + 1)/s_{n-2}$$

for all $n > 2$ (the Lyness recurrence). Then

$$\begin{aligned} s_3 &= \frac{y+1}{x}, \\ s_4 &= \frac{x+y+1}{xy}, \\ s_5 &= \frac{x+1}{y}, \\ s_6 &= x, \\ s_7 &= y, \end{aligned}$$

etc. (period 5).

(You can think of this as iteration of the map $(u, v) \mapsto (v, (v + 1)/u)$, or if you prefer homogeneous coordinates, $(u, v) \mapsto (v, (tv + t^2)/u)$.

Define $s_1 = x$, $s_2 = y$, and

$$s_n = \begin{cases} (s_{n-1} + 1)/s_{n-2} & \text{for } n \text{ odd,} \\ (s_{n-1}^2 + 1)/s_{n-2} & \text{for } n \text{ even} \end{cases}$$

for all $n > 2$. Then

$$\begin{aligned} s_3 &= \frac{y+1}{x}, \\ s_4 &= \frac{x^2+y^2+2y+1}{x^2y}, \\ s_5 &= \frac{x^2+y+1}{xy}, \\ s_6 &= \frac{x^2+1}{y}, \\ s_7 &= x, \\ s_8 &= y, \end{aligned}$$

etc. (period 6).

Likewise, the recurrence

$$s_n = \begin{cases} (s_{n-1}+1)/s_{n-2} & \text{for } n \text{ odd,} \\ (s_{n-1}^3+1)/s_{n-2} & \text{for } n \text{ even} \end{cases}$$

gives a sequence with period 8.

But the recurrence

$$s_n = \begin{cases} (s_{n-1}^a+1)/s_{n-2} & \text{for } n \text{ odd,} \\ (s_{n-1}^b+1)/s_{n-2} & \text{for } n \text{ even} \end{cases}$$

has non-periodic behavior for any pair (a,b) of positive integers other than $(1,1)$, $(1,2)$, $(2,1)$, $(1,3)$, and $(3,1)$ (Fomin and Zelevinsky, 2001).

Nevertheless, in each case, we get a sequence of Laurent polynomials, i.e., polynomials in x,y,x^{-1},y^{-1} (Fomin and Zelevinsky, 2001).

It also appears that all the coefficients of these Laurent polynomials are positive. (To see why this isn't trivial, note that a quotient of polynomials with positive coefficients need not have positive coefficients; e.g., $(x^3 + y^3)/(x + y) = x^2 - xy + y^2$.)

Positivity has been proved for the case $(a, b) = (2, 2)$. One proof makes use of lattice paths.

II. A simple non-periodic case

Consider the four-tiered graph

```
0   2   4   6   8  10
a   a   a   a   a   a
 \ / \ / \ / \ / \ / \
  b   b   b   b   b  ...
   \ / \ / \ / \ / \
    c   c   c   c  ...
     \ / \ / \ / \
      d   d   d  ...
```

For each directed path in the graph consisting of $2n$ rightward edges with initial and final vertices in the top tier, define the weight of the path to be the product of the labels of the $2n+1$ vertices on the path.

E.g., the directed path

```
0   2   4   6   8  10
a   a   a   a   a   a
 \                 /
  b   b   b   b   b
   \ / \         /
    c   c   c   c
         \ / \ /
      d   d   d
```

has weight $abcbcdcdcba = a^2b^3c^4d^2$.

Let $W(n)$ be the sums of the weights of all such paths of length $2n$.

Claim: For $n \geq 2$,

$$W(n)W(n-2) - W(n-1)^2 = a^{n+1}b^n c^{n-1} d^{n-2}.$$

In particular, if we put $a = x$, $b = y/x^2$, $c = x/y^2$, and $d = y$, we get $W(0) = x$, $W(1) = y$, and

$$W(n-1)W(n+1) - W(n)^2 = 1$$

for all $n \geq 1$, so that $W(n)$ is the n+1st term of the sequence x, y, $(y^2+1)/x$, $(y^4+2y^2+1+x^2)/x^2y$, $\ldots$ satisfying the recurrence

$$s_n = (s_{n-1}^2 + 1)/s_{n-2}.$$

Note that putting $x = y = 1$ gives the sequence $1, 1, 2, 5, 13, 34, \ldots$ (every second Fibonacci number).

The claim can be proved by a "reflection argument". If A is a path from 0 to $2n$ (in the top tier) and B is a path from 2 to $2(n-1)$, then typically A and B will cross, in which case they can swap edges so as to become a path A' from 0 to $2(n-1)$ and a path B' from 2 to $2n$. (Specifically, A and B swap all edges to the right of the rightmost point of intersection.)

This operation is weight-preserving if the weight of a pair (A, B) is defined as the product of the weights of A and B (and likewise for (A', B')).

The operation is reversible when applicable.

The one situation in which the operation is inapplicable is given by a particular pair of paths whose weights have product $a^{n+1}b^n c^{n-1} d^{n-2}$.

(This same principle, applied to larger collections of paths, leads to Lindstrom's Lemma on non-intersecting paths and determinants, rediscovered by Gessel and Viennot.)

The one-dimensional recurrence

$$s_n = (s_{n-1}^2 + 1)/s_{n-2}$$

with initial conditions $s_0 = x$, $s_1 = y$ can be understood as a special case of the two-dimensional recurrence

$$F(n,k) =$$

$$(F(n-1,k-1)F(n-1,k+1)+1)/F(n-2,k)$$

with initial conditions

$$\begin{aligned} F(0,k) &= x_k \\ F(1,k) &= y_k \end{aligned}$$

The one-dimensional recurrence corresponds to the case where $x_k = x$ and $y_k = y$ for all k.

(We restrict to the case $n \equiv k \pmod 2$ without loss of generality.)

This recurrence is naturally seen via an array

$$\begin{array}{ccccccccc}
F(0,0) & & F(0,2) & & F(0,4) & & F(0,6) & & F(0,8) \\
& F(1,1) & & F(1,3) & & F(1,5) & & F(1,7) & \\
& & F(2,2) & & F(2,4) & & F(2,6) & & \\
& & & F(3,3) & & F(3,5) & & & \\
& & & & F(4,4) & & & &
\end{array}$$

in which any four consecutive numbers forming the pattern

$$\begin{array}{ccc}
& A & \\
B & & C \\
& D &
\end{array}$$

satisfy $AD - BC = 1$, i.e.,

$$D = (BC+1)/A$$

(the "square recurrence").

E.g.,

$$\begin{aligned}
F(0,0) &= x_0, \\
F(1,1) &= y_1, \\
F(2,2) &= (y_1 y_3 + 1)/x_2, \\
F(3,3) &= (y_1 y_3^2 y_5 + y_1 y_3 + y_3 y_5 + 1 + x_2 x_4)/x_2 y_3 x_4,
\end{aligned}$$

etc.

Fact: $F(n,k)$ is always a Laurent polynomial.

Fact: The Laurent polynomial $F(n,k)$ has all coefficients equal to 1. Hence the number of terms in the numerator of $F(n,k)$ is equal to the value of $F(n,k)$ at $x = y = 1$, which is the $2n$th Fibonacci number, which counts lattice paths as described above.

In fact, there is a natural bijection between the terms of $F(n,k)$ and paths from vertex $n-k$ to vertex $n+k$ in the top tier of the graph.

Each term of $F(n,k)$ may be understood as the weight of the associated lattice path, under a suitable weighting scheme.

When one puts $x_i = x$ and $y_i = y$ for all i, one collapses together different terms, so that the coefficients are no longer all 1. Thus the Laurent polynomials $F(n,k)$ contain more combinatorial information than the Laurent polynomials s_n.

In fact, the Laurent polynomials $F(n,k)$ contain all the combinatorial information there is, since each of the objects being enumerated corresponds to its own Laurent monomial.

In this sense, the one-dimensional recurrence is "most naturally" viewed as a specialization of the two-dimensional recurrence.

Another combinatorial proof of the Laurentness of iterates of the mapping $(u, v) \mapsto (v, (v + 1)/u)$, and the positivity of the coefficients, can be given in terms of perfect matchings of the 2-by-$2n$ square grid:

```
o     o     o-----o     o     o-----o     o
|     |                 |                 |
|     |                 |                 |
o     o     o-----o     o     o-----o     o
```

(Perfect matchings of graphs are known in the physics literature as dimer states.)

Let $M(n)$ be the number of perfect matchings of the 2-by-$2n$ square grid. One can prove combinatorially that

$$M(n)M(n-2) = M(n-1)^2 + 1.$$

The two-dimensional recurrence

$$t_n = (t_{n-1}t_{n-2} + 1)/t_{n-3}$$

with initial conditions $t_1 = x$, $t_2 = y$, $t_3 = z$ can be understood using the exact same two-dimensional recurrence, but with different initial conditions:

$$\begin{array}{ccccccccccc}
 & & & & & & x & & y & & z \\
 & & & x & & y & & z & & ? & \\
x & & y & & z & & ? & & ? & & \\
 & z & & ? & & ? & & ? & & & \\
 & & ? & & ? & & ? & & & & \\
 & & & ? & & ? & & & & & \\
 & & & & ? & & & & & &
\end{array}$$

Moreover, one can show that the Laurent polynomials t_n give a weighted enumeration of paths in a shifted version of the grid.

So a higher-dimensional recurrence can unify many seemingly disparate lower-dimensional ones.

III. The octahedron recurrence

The Discrete Hirota equation

(see Zabrodin):

Let u, v, w be fixed vectors with the vector x varying over some fixed coset of $\mathbf{Z}u + \mathbf{Z}v + \mathbf{Z}w$.
Symmetric form:

$$aF(x+u)F(x-u)+bF(x+v)F(x-v)$$
$$+cF(x+w)F(x-w) = 0$$

Asymmetric form:

$$F(x+w)F(x-w) =$$
$$aF(x+u)F(x-u)+bF(x+v)F(x-v)$$

Note:

$$\begin{aligned}&(x+u)+(x-u)\\ &= (x+v)+(x-v)\\ &= (x+w)+(x-w).\end{aligned}$$

Rhombus tilings of hexagons

Let $H(a, b, c)$ be the number of tilings of an a, b, c, a, b, c semiregular hexagon (opposite sides of equal length, all internal angles equal to 120 degrees), using unit rhombuses. E.g., the tilings of

```
  *---*---*
 / \ / \ / \
*---*---*---*
 \ / \ / \ /
  *---*---*
```

are

```
  *---*---*       *---*---*       *---*---*
 /   /   / \     /   / \   \     / \   \   \
*---*---*   *   *---*   *---*   *   *---*---*
 \   \   \ /     \   \ /   /     \ /   /   /
  *---*---*       *---*---*       *---*---*
```

so $H(1, 2, 1) = 3$.

Note that $H(a, b, c)$ can also be interpreted as the number of perfect matchings of the dual graph (whose vertices correspond to the triangular cells, and whose edges correspond to adjacent pairs of triangular cells).

$H(a,b,c)$ satisfies the octahedron recurrences

$$H(a,b,c)H(a,b-1,c-1)$$
$$= H(a+1,b-1,c-1)H(a-1,b,c)$$
$$+H(a,b-1,c)H(a,b,c-1)$$

and

$$H(a,b,c)H(a,b,c-2)$$
$$= H(a,b,c-1)H(a,b,c-1)$$
$$-H(a-1,b+1,c-1)H(a+1,b-1,c-1).$$

Somos-4 and Somos-5 sequences

The Somos-4 sequence:

$$1, 1, 1, 1, 2, 3, 7, 23, 59, 314, 1529, 8209, 83313, \ldots$$

$$S_n S_{n-4} = S_{n-1} S_{n-3} + S_{n-2} S_{n-2}$$

The Somos-5 sequence:

$$1, 1, 1, 1, 1, 2, 3, 5, 11, 37, 83, 274, 1217, 6161, \ldots$$

$$S_n S_{n-5} = S_{n-1} S_{n-4} + S_{n-2} S_{n-3}$$

(These are related to elliptic divisibility sequences.)

It's known that the terms of these sequences are integers. Also, one can give exact formulas for the terms via elliptic theta functions.

Domino tilings of Aztec diamonds

Let $M(n)$ be the number of domino tilings of an "Aztec diamond of order n". E.g.,

```
        +---+---+
        |       |
    +---+---+---+---+
    |       |       |
+---+---+---+---+---+---+
|   |   |   |       |   |
+   +   +   +---+---+   +
|   |   |   |   |   |   |
+---+---+---+   +   +---+
    |       |   |   |
    +---+---+---+---+
        |       |
        +---+---+
```

is one of the 64 ways to tile an Aztec diamond of order 3 with 1-by-2 rectangles, so $M(3) = 64$.

Note that $M(n)$ can also be interpreted as the number of perfect matchings of the dual graph (whose vertices correspond to the square cells, and whose edges correspond to adjacent pairs of square cells).

```
         -------
        |       |                             o---o
     ---------------
    |       |       |                     o---o   o---o
 ---|-------|-------|---
|   |   |   |       |   |             o   o   o   o---o   o
|   |   |   |-------|   |   <--->     |   |   |           |
|   |   |   |   |   |   |             o   o   o   o   o   o
 ---|-------|   |   |---                          |   |
    |       |   |   |                     o---o   o   o
     -----------|---
        |       |                             o---o
         -------

    "domino tiling of                 "perfect matching of an
    an Aztec diamond"                   Aztec diamond graph"
```

$M(n)$ satisfies the octahedron recurrence

$$M(n)M(n-2) =$$
$$M(n-1)M(n-1)+M(n-1)M(n-1).$$

(In fact, $M(n) = 2^{n(n+1)/2}$.)

IV. Condensation of determinants

$$M = \begin{pmatrix} m_{1,1} & m_{1,2} & \dots & m_{1,n-1} & m_{1,n} \\ m_{2,1} & m_{2,2} & \dots & m_{2,n-1} & m_{2,n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ m_{n-1,1} & m_{n-1,2} & \dots & m_{n-1,n-1} & m_{n-1,n} \\ m_{n,1} & m_{n,2} & \dots & m_{n,n-1} & m_{n,n} \end{pmatrix}$$

$$M_C = \begin{pmatrix} m_{2,2} & \dots & m_{2,n-1} \\ \vdots & \ddots & \vdots \\ m_{n-1,2} & \dots & m_{n-1,n-1} \end{pmatrix} \quad (\text{“center”})$$

$$M_{TL} = \begin{pmatrix} m_{1,1} & m_{1,2} & \dots & m_{1,n-1} \\ m_{2,1} & m_{2,2} & \dots & m_{2,n-1} \\ \vdots & \vdots & \ddots & \vdots \\ m_{n-1,1} & m_{n-1,2} & \dots & m_{n-1,n-1} \end{pmatrix} \quad (\text{“top left”})$$

with the $n-1$ by $n-1$ minors M_{TR} (“top right”), M_{BL} (“bottom left”), and M_{BR} (“bottom right”) defined similarly.

The Desnanot-Jacobi identity

$$\det(M)\ \det(M_C) =$$

$$\det(M_{TL})\ \det(M_{BR}) - \det(M_{TR})\ \det(M_{BL})$$

Dodgson condensation

To compute the determinant of an n-by-n matrix, iteratively use this identity to compute the determinants of the connected minors of orders 1, 2, ..., n. (A "connected" k-by-k minor is formed by taking k consecutive rows and k consecutive columns.)

Note: a minor of order 0 has determinant 1.

Example:

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 4 \\ 1 & 6 & 36 \\ 1 & 12 & 144 \end{pmatrix} \to$$

$$\begin{pmatrix} 2 & 12 \\ 2 & 48 \end{pmatrix} \to \begin{pmatrix} 12 \end{pmatrix}$$

Dodgson pyramids

Stacking the matrices in layers, we get a relation of discrete Hirota type, relating the values associated with the vertices of an octahedron:

```
                                a
                             b------c
Rule: For the 3D pattern    /      /  ,
                           d------e
                                f
      we have af = be - cd.
```

V. Alternating-sign matrices

For $1 \leq m \leq n-1$, the determinant of the n-by-n matrix M can be expressed as a rational function of the determinants of the $(n-m)$-by-$(n-m)$ and $(n-m-1)$-by-$(n-m-1)$ connected minors of M.

(We can arrange these determinants in the form of an $(m+1)$-by-$(m+1)$ and an $(m+2)$-by-$(m+2)$ matrix; we can superimpose these two matrices, obtaining a "bimatrix".)

Theorem (Robbins and Rumsey, 1986): This rational function (the "determinant" of the bimatrix) is formally a Laurent polynomial in the entries of the bimatrix (the determinants of the connected minors of order $n-m$ and $n-m-1$).

$m = 1$:

$$\ldots \to \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \to \begin{pmatrix} j & k \\ l & m \end{pmatrix} \to \begin{pmatrix} X \end{pmatrix} \ ,$$

$$\begin{aligned} X &= (jm - kl)/e \\ &= j^1 m^1 e^{-1} - k^1 l^1 e^{-1} \\ &= e^{-1} j^1 k^0 l^0 m^1 - e^{-1} j^0 k^1 l^1 m^0 \\ &= \left(\begin{pmatrix} 1 & & 0 \\ & -1 & \\ 0 & & 1 \end{pmatrix} \right) - \left(\begin{pmatrix} 0 & & 1 \\ & -1 & \\ 1 & & 0 \end{pmatrix} \right) \ . \end{aligned}$$

$m = 2$:

$$\ldots \to \begin{pmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{pmatrix} \to \begin{pmatrix} q & r & s \\ t & u & v \\ w & x & y \end{pmatrix} \to$$

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \to \begin{pmatrix} X \end{pmatrix} \ ,$$

$m = 2$ (continued):

X = an alternating sum of eight Laurent monomials

$$
\begin{aligned}
= {} & \left(\begin{pmatrix} 1 & & 0 & & 0 \\ & -1 & & 0 & \\ 0 & & 1 & & 0 \\ & 0 & & -1 & \\ 0 & & 0 & & 1 \end{pmatrix}\right) - \left(\begin{pmatrix} 0 & & 1 & & 0 \\ & -1 & & 0 & \\ 1 & & 0 & & 0 \\ & 0 & & -1 & \\ 0 & & 0 & & 1 \end{pmatrix}\right) \\
& - \left(\begin{pmatrix} 1 & & 0 & & 0 \\ & -1 & & 0 & \\ 0 & & 0 & & 1 \\ & 0 & & -1 & \\ 0 & & 1 & & 0 \end{pmatrix}\right) - \left(\begin{pmatrix} 0 & & 0 & & 1 \\ & 0 & & -1 & \\ 0 & & 1 & & 0 \\ & -1 & & 0 & \\ 1 & & 0 & & 0 \end{pmatrix}\right) \\
& + \left(\begin{pmatrix} 0 & & 1 & & 0 \\ & 0 & & -1 & \\ 0 & & 0 & & 1 \\ & -1 & & 0 & \\ 1 & & 0 & & 0 \end{pmatrix}\right) + \left(\begin{pmatrix} 0 & & 0 & & 1 \\ & 0 & & -1 & \\ 1 & & 0 & & 0 \\ & -1 & & 0 & \\ 0 & & 1 & & 0 \end{pmatrix}\right) \\
& + \left(\begin{pmatrix} 0 & & 1 & & 0 \\ & -1 & & 0 & \\ 1 & & -1 & & 1 \\ & 0 & & -1 & \\ 0 & & 1 & & 0 \end{pmatrix}\right) - \left(\begin{pmatrix} 0 & & 1 & & 0 \\ & 0 & & -1 & \\ 1 & & -1 & & 1 \\ & -1 & & 0 & \\ 0 & & 1 & & 0 \end{pmatrix}\right).
\end{aligned}
$$

Theorem (Robbins and Rumsey, 1986, continued):
Moreover, "Laurentness" continues to hold if the Dodgson recurrence

$$af = be - cd$$

is replaced by the recurrence

$$af = be + cd$$

or the more general

$$af = be + \lambda cd.$$

This is called the λ-determinant of the bimatrix.

Theorem (Robbins and Rumsey, 1986, continued):
The λ-determinant of a bimatrix is a sum of

$$2^{m(m+1)/2}$$

monomials, in which each coefficient is plus or minus a power of λ, and the exponents of all the variables equal $+1$, 0, or -1. The terms correspond to the compatible pairs of ASMs (Alternating-Sign Matrices) of order m and $m+1$.

Compatibility

The ASMs of order 3, and the ASMs of order 2 they are respectively compatible with:

```
(1 0 0) (0 1 0) (1 0 0) (0  1 0) (0 0 1) (0 1 0) (0 0 1)
(0 1 0) (1 0 0) (0 0 1) (1 -1 1) (1 0 0) (0 0 1) (0 1 0)
(0 0 1) (0 0 1) (0 1 0) (0  1 0) (0 1 0) (1 0 0) (1 0 0)
      \       \    |       / \       |    /       /
         \     \   |     /     \     |   /     /
            \   \  |   /         \   |  /   /
               \ \ | /             \ | / /
                (1 0)               (0 1)
                (0 1)               (1 0)
```

VI. ASMs and Aztec diamonds

Grensing, Carlsen, and Zapp (1980) conjectured that for graphs of the form

```
                o-----o
                |     |
                |     |
          o-----o-----o-----o
          |     |     |     |
          |     |     |     |
    o-----o-----o-----o-----o-----o
    |     |     |     |     |     |
    |     |     |     |     |     |
    o-----o-----o-----o-----o-----o
          |     |     |     |
          |     |     |     |
          o-----o-----o-----o
                |     |
                |     |
                o-----o
```

the number of perfect matchings (aka dimer configurations) is $2^{n(n+1)/2}$, where the rows and columns have respective lengths $2, 4, 6, \ldots, 2n, 2n, \ldots, 6, 4, 2$.

This was proved by Elkies, Kuperberg, Larsen, and Propp (1992), using the language of tilings instead of dimer covers.

But in a sense the result was proved by Robbins and Rumsey (1986) using the language of compatible pairs of ASMs.

From matchings to pairs of ASMs

(Elkies, Kuperberg, Larsen, Propp)

In each hole, write 1 minus the number of neighboring edges included in the matching.

```
0   o   0   o   1   o   0
  /       /           \
o   0   o  -1   o   0   o
              /
0   o   1   o   0   o   0
  /                   \
o   0   o   0   o  -1   o
          \       \
1   o  -1   o   0   o   1
      \
o  -1   o   0   o   0   o
  \           /       /
0   o   1   o   0   o   0
```

Decompose the bimatrix into two matrices, and negate the entries in the smaller matrix.

From pairs of ASMs to matchings (Carroll)

Insert 0's into the holes in the bimatrix, and at each location, record the (possibly empty) sum of all the entries above and/or to the left of it, obtaining the "corner-sum matrix" of the bimatrix.

```
0     0     0     0     0     0     0     0
         o           o           o
0     0     0     0     0     1     1     1
   o           o           o           o
0     0     0     0    -1     0     0     0
         o           o           o
0     0     0     1     0     1     1     1
   o           o           o           o
0     0     0     1     0     1     0     0
         o           o           o
0     1     1     1     0     1     0     1
   o           o           o           o
0     1     0     0    -1     0    -1     0
         o           o           o
0     1     0     1     0     1     0     1
```

Then replace each corner-sum by a mark, as prescribed by the following table, based on the dimensions of the sub-array and the value of the sum.

Even-by-even block:
  corner sum is -1: edge (/)
  corner sum is 0: no edge
  corner sum is 1: **incompatible**
Even-by-odd or odd-by-even block:
  corner sum is -1: **incompatible**
  corner sum is 0: no edge
  corner sum is 1: edge (\)
Odd-by-odd block:
  corner sum is -1: **incompatible**
  corner sum is 0: edge (/)
  corner sum is 1: no edge

The generic Dodgson pyramid with formal indeterminates at levels -1 and 0:

Level 1: Laurent polynomials with 2 terms
Level 2: Laurent polynomials with 8 terms
Level 3: Laurent polynomials with 64 terms
... etc.

Specialize!

(i.e., turn the Laurent polynomials into numbers)

The numbers $1, 1, 2, 8, 64, \ldots$ (of the form $2^{n(n+1)/2}$) yield a solution to the discrete Hirota equation, with initial conditions in a slab made of two planes of 1's:

```
1    1    1    1    1
                             1    1    1    1
1    1    1    1    1                                2    2    2
                             1    1    1    1                           8    8
1    1    1    1    1                                2    2    2                     64
                             1    1    1    1                           8    8
1    1    1    1    1                                2    2    2
                             1    1    1    1
1    1    1    1    1
```

If we let $M(n)$ be the number of perfect matchings of the Aztec diamond graph of order n, then we obtain

$$M(n+1)M(n-1) =$$
$$M(n)M(n) + M(n)M(n) \ ,$$

a 1-dimensional discrete Hirota equation that gives us an inductive proof of the formula $M(n) = 2^{n(n+1)/2}$.

In 1997 Eric Kuo found a direct combinatorial proof of this formula (via "graphical condensation").

VII. Somos sequences

Theorem (Somos; Fomin and Zelevinsky): If we put $S(1) = w$, $S(2) = x$, $S(3) = y$, and $S(4) = z$, and

$$S(n) = \frac{S(n-1)S(n-3) + S(n-2)^2}{S(n-4)}$$

for $n > 4$, then $S(n)$ is a Laurent polynomial in w, x, y, z (i.e., a polynomial in $w, w^{-1}, x, x^{-1}, y, y^{-1}, z, z^{-1}$).

Theorem (Somos; Fomin and Zelevinsky): If we put $S(1) = 1$, $S(2) = 1$, $S(3) = 1$, and $S(4) = 1$, and

$$S(n) = \frac{uS(n-1)S(n-3) + vS(n-2)^2}{S(n-4)}$$

for $n > 4$, then $S(n)$ is a polynomial in u, v.

Theorem (Fomin and Zelevinsky): Fix positive integers a, b, d with $a+b<d$. Put $S(i) = x_i$ for $i = 1, 2, ..., d$ and put

$$S(n) = \frac{uS(n-a)S(n-d+a) + vS(n-b)S(n-d+b)}{S(n-d)}$$

for $n > d$. Then $S(n)$ is polynomial in $x_1, x_1^{-1}, ..., x_d, x_d^{-1}, u, v$.

Special cases:

$d = 4, a = 1, b = 2$: Somos-4
$d = 5, a = 1, b = 2$: Somos-5

Quasi-theorem (Propp, Bousquet-Mélou and West, 2002):

All the coefficients in this polynomial are positive integers.

The values of Somos-4 also yield an intrinsically one-dimensional solution to the discrete Hirota equation, with initial conditions in a tilted slab made of four tilted planes of 1's:

```
   1   2   3   7
                       3   7  23
   1   1   2   3                      23  59
                       2   3   7                     314
   1   1   1   2                       7  23
                       1   2   3
   1   1   1   1
```

Un-specialize! (???)

(i.e., turn the numbers back into Laurent polynomials)

Implementing recurrences in MAPLE

```
f := proc(n,i,j) option remember;
    if n<0 then undefined
    elif n<4 then x(n,i,j);
    else simplify( ( f(n-1,i-1,j)*f(n-3,i+1,j)
                   + f(n-2,i,j-1)*f(n-2,i,j+1))
                 /  f(n-4,i,j)); fi; end;
```

Empirically, one finds that `f(n,i,j)` is a Laurent polynomial in the x-variables in which every coefficient equals $+1$, so that the number of terms in `f(n,i,j)` is equal to the result of specializing all the x's to equal 1, which is equal to the nth term of the Somos-4 sequence.

The Somos-4 graphs

```
                                             o--o
                                             |  |
                                          o--o--o--o
                                          |  |  |  |
                           o--o        o--o  o--o  o--o
                           |  |        |  |  |  |  |  |
                        o--o  o--o     o  o--o  o--o  o
                        |  |  |  |        |  |  |  |
            o--o        o--o--o--o        o--o--o--o     . . .
            |  |        |  |  |  |        |  |  |  |
         o--o  o--o     o--o  o--o        o--o  o--o
         |  |  |  |     |  |  |  |        |  |  |  |
o--o     o  o--o  o     o  o--o  o        o  o--o  o
|  |        |  |           |  |              |  |
o--o        o--o           o--o              o--o
|  |        |  |           |  |              |  |
o--o        o--o           o--o              o--o
|  |        |  |           |  |              |  |
o  o        o  o           o  o              o  o
```

VIII. The cube recurrence

Theorem (Fomin and Zelevinsky): Fix positive integers a, b, c and let $a' = b + c$, $b' = a + c$, and $c' = a + b$. Put $S(i) = x_i$ for $i = 1, 2, ..., a + b + c$ and put

$$S(n) = \frac{uS(n-a)S(n-a') + vS(n-b)S(n-b') + wS(n-c)S(n-c')}{S(n-a-b-c)}$$

for $n > a+b+c$. Then $S(n)$ is polynomial in $x_1, x_1^{-1}, ..., x_{a+b+c}, x_{a+b+c}^{-1}$, u, v, w.

Special cases:

$a = 1, b = 2, c = 3$: Somos-6

$$S(n)S(n-6) = S(n-1)S(n-5)$$
$$+S(n-2)S(n-4) + S(n-3)^2$$

$a = 1, b = 2, c = 4$: Somos-7

$$S(n)S(n-7) = S(n-1)S(n-6)$$
$$+S(n-2)S(n-5) + S(n-3)S(n-4)$$

(Somos-6 and Somos-7 were first proved to give integers by Dean Hickerson.)

The cube recurrence (or discrete Miwa equation)

```
                                  a
                                / | \
                              /   |   \
                            /     |     \
                          b       c       d
                          | \   / | \   / |
Rule: For the 3D pattern  |   X       X   | ,
                          | /   \ | /   \ |
                          e       f       g
                            \     |     /
                              \   |   /
                                \ | /
                                  h

      we have ah = bg + cf + de.
```

Conjecture (Propp, 1998):
For $i+j+k = -1, 0$, or 1, let $S(i,j,k) = x_{i,j,k}$, and for $i+j+k > 1$ inductively define $S(i,j,k)$ by the cube recurrence

$$\begin{aligned} & S(i,j,k)S(i-1,j-1,k-1) \\ = \; & S(i-1,j,k)S(i,j-1,k-1) \\ & +S(i,j-1,k)S(i-1,j,k-1) \\ & +S(i,j,k-1)S(i-1,j-1,k). \end{aligned}$$

Then for all i,j,k with $i+j+k \geq -1$, $S(i,j,k)$ is a Laurent polynomial in the x-variables, with all coefficients equal to 1 and all exponents bounded between -1 and 4 (inclusive).

The Laurentness part of the claim was proved by Fomin and Zelevinsky. The rest was proved by my students Gabriel Carroll and David Speyer (with some input from me) in Spring 2002, when they discovered that these Laurent polynomials are encoding a hitherto unstudied kind of combinatorial object, which they call a *grove*.

A grove of order n is a special kind of forest on the triangle graph with $n+1$ vertices on a side. It is required that:

- every vertex is joined to a point on the boundary by a path; and
- two boundary-vertices v, w are joined by a path iff they are related by a reflection through a median that passes through a corner-point P of the triangle with the property that no corner of the triangle is closer to v or w than P.

(It follows that the corner vertices are isolated, and that every vertex other than the three corner vertices belongs to at least one edge.)

A grove of order 5

```
          o

        o---o

      o   o---o
       \ /
    o   o   o   o
     \     /   /
  o   o---o   o   o
   \     /   /   /
o   o   o   o   o   o
```

A grove of order 6

```
            o

          o---o
           \
        o   o   o
         \     /
      o   o---o   o
       \         /
    o   o---o---o   o
     \       \     /
  o   o---o   o   o   o
   \     /   /   /   /
o   o   o   o   o   o   o
```

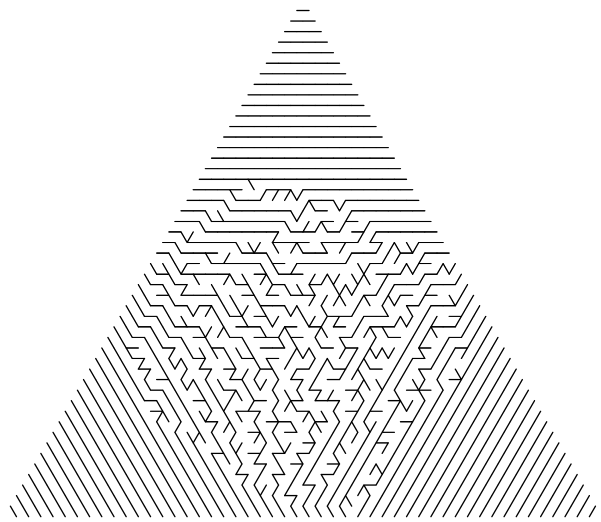
Theorem (Carroll and Speyer, 2002):
The number of groves of order m is

$$3^{\lfloor m^2/4 \rfloor},$$

and these correspond to the monomials in $S(i,j,k)$, where $m = i+j+k$.

We knew the RHS ahead of time; it was the LHS that was mysterious!

In[19]:= **Grove[50]**



Out[19]= - Graphics -

A random grove

IX. An integrable system

Continuous Painlevé II:

$$y''(x) = 2y^3 + xy + c$$

Discrete version:

$$nx_n + t(1 - x_n^2)(x_{n+1} + x_{n-1}) = 0$$

Autonomous discrete version:

$$x_n + t(1 - x_n^2)(x_{n+1} + x_{n-1}) = 0$$

Replace x_n by ix_n, so that the recurrence becomes

$$x_n + t(1 + x_n^2)(x_{n+1} + x_{n-1}) = 0.$$

Let $u = x_{-1}$ and $v = x_0$. Then the sequence can also be generated by the recurrence

$$x_{n+1}x_{n-1} = (x_n^2 + d)/(x_n^2 + 1)$$

where

$$d = -u^2 - v^2 - u^2v^2 - uv/t.$$

If we treat d as a formal variable, x_n (with $n \geq 1$) can be written as

$$P_n(u, v, d)/Q_n(u, v, d)$$

where P_n, Q_n are Laurent polynomials whose degrees grow subexponentially (quadratically, in fact).

P_n and Q_n satisfy a pair of joint (1D) discrete Hirota equations

$$P_{n+1}P_{n-1} = P_nP_n + dQ_nQ_n$$

$$Q_{n+1}Q_{n-1} = P_nP_n + Q_nQ_n$$

with initial conditions

$$P_{-1} = u, \; P_0 = v,$$

$$Q_{-1} = 1, \; Q_0 = v.$$

The one-dimensional family of polynomials P_n, Q_n can be lifted to a three-dimensional family of Laurent polynomials $P(n,i,j)$, $Q(n,i,j)$ satisfying a pair of joint 3D discrete Hirota equations:

$$P(n+1,i,j)P(n-1,i,j) =$$
$$P(n,i-1,j)P(n,i+1,j)+$$
$$dQ(n,i,j-1)Q(n,i,j+1)\ ,$$

$$Q(n+1,i,j)Q(n-1,i,j) =$$
$$P(n,i-1,j)P(n,i+1,j)+$$
$$Q(n,i,j-1)Q(n,i,j+1)$$

The Laurent polynomials $P(n,i,j)$, $Q(n,i,j)$ enumerate weighted perfect matchings of the order n Aztec diamond graph.

X. Markov numbers

The Markov numbers are solutions to the equation $a^2 + b^2 + c^2 = 3abc$, e.g., $(a, b, c) = (1, 1, 1)$, $(1, 1, 2)$, $(1, 2, 5)$, $(1, 2, 13)$, $(2, 5, 29)$.

If (a, b, c) is a solution, so is (a, b, c') with $c' = (a^2+b^2)/c$. Ditto for (a, b', c) and (a', b, c) where $b' = (a^2+c^2)/b$ and $a' = (b^2 + c^2)/a$.

These are examples of “cluster moves” (in the cluster algebra framework of Fomin and Zelevinsky).

It is not hard to show that every triple of Markov numbers is obtainable from $(1, 1, 1)$ by cluster moves.

Fomin and Zelevinsky (2001) showed that every triple obtainable from (x, y, z) by cluster moves is a Laurent polynomial in the variables x, y, z.

Gregg Musiker, Rui Viana, and other students of mine involved in REACH (Research Experiences in Algebraic Combinatorics at Harvard) have shown that the coefficients of these Laurent polynomials are positive. There proof gives a combinatorial interpretation to all of the coefficients in terms of perfect matchings of suitable graphs. See links available at

`www.math.harvard.edu/~propp/reach`

XI. Conclusions

XII. Bibliography

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See also links accessible from the following web-pages:
`www.math.wisc.edu/~propp/somos.html`
`www.math.wisc.edu/~propp/bilinear.html`
`www.math.harvard.edu/~propp/reach`