Farey-ish fractions from weighted mediants

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CCCC L, April 3, 2011

It's well-known that if you start with the list containing just the two fractions $\frac{0}{1}$ and $\frac{1}{1}$ and successively interpose the mediant $\frac{a+c}{b+d}$ between successive elements $\frac{a}{b}$ and $\frac{c}{d}$ of the list, every rational between 0 and 1 will eventually appear.

(a) Consider a variant of this in which we interpose the weighted mediants $\frac{2a+c}{2b+d}$ and $\frac{a+2c}{b+2d}$ instead, reducing common factors as they appear, like this:

$$\begin{array}{c}
\frac{0}{1}\ \frac{1}{1} \\[1em]
\frac{0}{1}\ \frac{1}{3}\ \frac{2}{3}\ \frac{1}{1} \\[1em]
\frac{0}{1}\ \frac{1}{5}\ \frac{2}{7}\ \frac{1}{3}\ \frac{4}{9}\ \frac{5}{9}\ \frac{2}{3}\ \frac{5}{7}\ \frac{4}{5}\ \frac{1}{1} \\[1em]
\frac{0}{1}\ \frac{1}{7}\ \frac{2}{11}\ \frac{1}{5}\ \cdots\ \frac{1}{3}\ \frac{2}{5}\ \frac{3}{7}\ \frac{4}{9}\ \cdots\ \frac{4}{5}\ \frac{9}{11}\ \frac{6}{7}\ \frac{1}{1} \\[1em]
\vdots
\end{array}$$

Note that when we interpose weighted mediants between $\frac{1}{3}$ and $\frac{4}{9}$, obtaining $\frac{2\cdot 1+4}{2\cdot 3+9} = \frac{6}{15}$ and $\frac{1+2\cdot 4}{3+2\cdot 9} = \frac{9}{21}$, we reduce $\frac{6}{15}$ to $\frac{2}{5}$ and $\frac{9}{21}$ to $\frac{3}{7}$ before entering them in the table.

Does every rational between 0 and 1 with odd denominator eventually appear in this table?

I've checked that for all n up through 15, the nth row of the table (counting the top row as the 0th) contains all rationals in $[0,1]$ with denominator $2n+1$.

(In many ways it's more natural *not* to remove common factors as they appear; then, as Bill Thurston pointed out, we're just looking at the orbits of two vectors under the semigroup generated by three 2-by-2 integer matrices, and most proper fractions with odd denominator won't occur. But that's not the game we're playing here.)

(b) Suppose that one instead forms an array by interposing the weighted mediants $\frac{4a+c}{4b+d}$, $\frac{3a+2c}{3b+2d}$, $\frac{2a+3c}{2b+3d}$, and $\frac{a+4c}{b+4d}$ between previously adjacent fractions $\frac{a}{b}$ and $\frac{c}{d}$ (reducing each fraction that appears to lowest terms before entering it). Then one does *not* obtain every rational between 0 and 1 with odd denominator; for instance, $\frac{1}{3}$ never appears.

I find that every fraction in the 10th row of the table has denominator congruent to 1 mod 4, and most fractions of this form with small denominator do occur althought a few don't; specifically, $\frac{4}{13}$, $\frac{6}{13}$, $\frac{7}{13}$, and $\frac{9}{13}$ haven't occurred by level 10 even though all the other proper fractions with denominator 13 occurred back in level 4 or earlier. The only other proper fractions with denominator congruent to 1 mod 4 and less than 30 that haven't occurred by level 10 are $\frac{4}{29}$ and $\frac{25}{29}$.

Can we characterize those rationals that appear in this table?

In particular, can we show that only fractions with denominators congruent to 1 mod 4 occur? The set of such fractions is not closed under taking weighted mediants (e.g., one weighted mediant of $\frac{1}{9}$ and $\frac{2}{9}$ is $\frac{4\cdot 1+1\cdot 2}{4\cdot 9+1\cdot 9}=\frac{6}{45}=\frac{2}{15}$), so the claim is not completely trivial.