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BRIEF COMMUNICATIONS

On the Laurent Phenomenon for Somos-4 and Somos-5 Sequences

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Abstract. In this paper we strengthen the result of Fomin and Zelevinsky (2002) on the Laurent phenomenon for Somos-4 and Somos-5 sequences.

Key words: Somos sequence, elliptic function, addition theorem, Laurent phenomenon.

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1. Introduction. A *Somos-k sequence* is a sequence $\{s_n\}$ satisfying a kth order ($k \geqslant 2$) quadratic recurrence relation of the form

$$s_{n+k}s_n = \sum_{1\leqslant j\leqslant k/2} \alpha_j s_{n+k-j}s_{n+j}, \tag{1}$$

where the α_j ($1 \leqslant j \leqslant k/2$) are constants.

One distinguishes the important class of Somos sequences that have the Laurent property; i.e., all terms are *Laurent polynomials* in the initial conditions, $s_n \in \mathbb{Z}[s_1^{\pm 1}, \dots, s_n^{\pm 1}, \alpha_1, \dots, \alpha_{\lfloor k/2\rfloor}]$.

The Laurent property of the Somos-2 and Somos-3 sequences follows from the explicit formulas

$$s_n = \alpha_1^{n(n-1)/2} s_0^{1-n} s_1^n$$

for $k = 2$ and

$$s_n = \begin{cases} \alpha_1^{n^2/4} s_{-1}^{-n/2} s_0 s_1^{n/2} & \text{if } n \text{ is even},\\ \alpha_1^{(n^2-1)/4} s_{-1}^{(1-n)/2} s_1^{(n+1)/2} & \text{if } n \text{ is odd} \end{cases}$$

for $k = 3$. There are no such simple formulas for $k \geqslant 4$. Based on the theory of cluster algebras, Fomin and Zelevinsky [3] proved the Laurent property of the Somos-k sequence for $k = 4, 5, 6, 7$. In particular, it follows that the Somos-k sequences ($k = 4, 5, 6, 7$) are integer-valued for $s_1 = \dots = s_k = \alpha_1 = \dots = \alpha_{\lfloor k/2\rfloor} = 1$. In the present paper, we prove a stronger version of this statement for $k = 4, 5$.

Theorem 1. *Define a Somos-*4 *sequence by the initial conditions* $s_{-1} = u$, $s_0 = x$, $s_1 = y$, *and* $s_2 = v$ *and the recurrence relation*

$$s_{n+2}s_{n-2} = \alpha s_{n+1}s_{n-1} + \beta s_n^2, \tag{2}$$

where $\alpha = uvxyw$, $\beta = uvxyz$, *and* u, v, w, x, y, *and* z *are independent formal variables. Then*

$$s_n \in \mathbb{Z}[u, v, w, x, y, z].$$

Theorem 2. *Define a Somos-*5 *sequence by the initial conditions* $s_{-2} = u$, $s_{-1} = x$, $s_0 = t$, $s_1 = y$, *and* $s_2 = v$ *and the recurrence relation*

$$s_n s_{n-5} = \alpha s_{n-1}s_{n-4} + \beta s_{n-2}s_{n-3}, \tag{3}$$

where $\alpha = uvxyw$, $\beta = uvxyz$, *and* t, u, v, w, x, y, *and* z *are independent formal variables. Then* $s_n \in \mathbb{Z}[t, u, v, w, x, y, z]$.

Remark 1. Special cases of Theorem 1 were considered earlier by Somos [6] ($w = z$ and $x = y = 1$) and Monina [1] ($x = y = 1$).

2. Somos-4. For this sequence $\{s_n\}_{n=-\infty}^{\infty}$, we define the matrices

$$M_s^{(0)} = (s_{m+n}s_{m-n})_{m,n=-\infty}^{\infty}, \qquad M_s^{(1)} = (s_{m+n+1}s_{m-n})_{m,n=-\infty}^{\infty}$$

By

$$M_s^{(0)}\begin{pmatrix} m_1,\dots,m_k \\ n_1,\dots,n_k \end{pmatrix} \quad \text{and} \quad M_s^{(1)}\begin{pmatrix} m_1,\dots,m_k \\ n_1,\dots,n_k \end{pmatrix}$$

we denote the finite submatrices of $M_s^{(0)}$ and $M_s^{(1)}$, respectively, formed by the entries at the intersections of rows $m_1,\dots,m_k$ and columns $n_1,\dots,n_k$.

Set

$$D_s^{(j)}\begin{pmatrix} m_1,\dots,m_k \\ n_1,\dots,n_k \end{pmatrix} = \det M_s^{(j)}\begin{pmatrix} m_1,\dots,m_k \\ n_1,\dots,n_k \end{pmatrix}, \qquad j=0,1.$$

A key property of Somos-4 sequences is that the rank of the matrices $M_s^{(j)}$, $j=0,1$, does not exceed 2 (and is 2 in general position). This follows, for example, from the general formula

$$s_n = AB^n\frac{\sigma(z_0+nz)}{\sigma(z)^{n^2}} \tag{4}$$

expressing the elements of the sequence via the Weierstrass sigma function (see [8] and [4]).

For elementary proofs, see [7] and [2].

Theorem 3. *Let $\{s_l\}$ be an arbitrary Somos-4 sequence. Then*

$$D_s^{(j)}\begin{pmatrix} m_1,m_2,m_3 \\ n_1,n_2,n_3 \end{pmatrix} = 0, \qquad j=0,1, \tag{5}$$

for any integers m_1, m_2, m_3 and n_1, n_2, n_3.

Proof of Theorem 1. It follows from the relations

$$s_3 = xyv(y^2w+xvz), \quad s_{-2} = xyu(x^2w+yuz), \quad s_{-3} = xu^2v(x^4ywz+x^2y^2uz^2+uw)$$

that the assertion of the theorem holds for the numbers l with $|l|\leqslant 3$. Therefore, we assume that $|l|>3$ in what follows. We will prove the theorem by induction, assuming that the desired statement has already been proved for numbers with absolute value less than $|l|$.

For even numbers $l=2n$ ($|n|\geqslant 2$), we use the relation

$$D_s^{(0)}\begin{pmatrix} n,1,0 \\ n,1,0 \end{pmatrix} = \begin{vmatrix} s_{2n}x & s_{n+1}s_{n-1} & s_n^2 \\ s_{1+n}s_{1-n} & xv & y^2 \\ s_ns_{-n} & yu & x^2 \end{vmatrix} = 0, \tag{6}$$

which is a special case of Eq. (5). This relation can be reduced by equivalence transformations to the form

$$x(x^3v-y^3u)s_{2n} = \begin{vmatrix} xvs_n^2 - y^2s_{n+1}s_{n-1} & s_{1+n}s_{1-n} \\ yus_n^2 - x^2s_{n+1}s_{n-1} & s_ns_{-n} \end{vmatrix}. \tag{7}$$

Let us show that the resulting determinant is divisible without remainder by x^3v-y^3u. Consider the right-hand side of Eq. (7) as a polynomial in the variable v. Dividing it with remainder by x^3v-y^3u, we obtain the relation

$$x(x^3v-y^3u)s_{2n} = (x^3v-y^3u)q(u,v,w,x,y,z) + r(u,w,x,y,z), \tag{8}$$

where $q(u,v,w,x,y,z)\in\mathbb{Z}[u,v,w,x^{\pm1},y,z]$ and $r(u,w,x,y,z)\in\mathbb{Z}[u,w,x^{\pm1},y,z]$. For positive initial conditions u, v, x, and y and positive values of the parameters w and z, the recurrence relation (2) defines a two-way infinite positive sequence $\{s_n\}$. In particular, all elements of this sequence are well defined. (Division by zero never occurs when computing these elements.) Therefore, by setting $v=y^3u/x^3$ in (7), we find that the remainder $r(u,w,x,y,z)$ is identically zero. Thus, $xs_{2n}\in\mathbb{Z}[u,v,w,x^{\pm1},y,z]$.

Considering the right-hand side of (7) as a polynomial in the variable u, we find in a similar way that $xs_{2n} \in \mathbb{Z}[u, v, w, x, y^{\pm 1}, z]$. Therefore, $xs_{2n} \in \mathbb{Z}[u, v, w, x, y, z]$. If we reproduce the entire argument replacing the determinant (6) with the determinant

$$D_s^{(0)} \begin{pmatrix} n+1, 1, 0 \\ n-1, 1, 0 \end{pmatrix} = \begin{vmatrix} s_{2n} v & s_{n+2} s_n & s_{n+1}^2 \\ s_n s_{2-n} & xv & y^2 \\ s_{n-1} s_{1-n} & yu & x^2 \end{vmatrix} = 0,$$

then we obtain $vs_{2n} \in \mathbb{Z}[u, v, w, x, y, z]$. Thus, $s_{2n} \in \mathbb{Z}[u, v, w, x, y, z]$.

For odd $l = 2n + 1$ ($-2 \leqslant n \leqslant 1$), the assertion of the theorem can be proved in a similar way based on the relations

$$D_s^{(0)} \begin{pmatrix} n+1, 1, 0 \\ n, 1, 0 \end{pmatrix} = \begin{vmatrix} s_{2n+1} y & s_{n+2} s_n & s_{n+1}^2 \\ s_{1+n} s_{1-n} & xv & y^2 \\ s_n s_{-n} & yu & x^2 \end{vmatrix} = 0,$$

$$D_s^{(0)} \begin{pmatrix} n, 1, 0 \\ n+1, 1, 0 \end{pmatrix} = \begin{vmatrix} s_{2n+1} x & s_{n+1} s_{n-1} & s_n^2 \\ s_{n+2} s_{-n} & xv & y^2 \\ s_{n+1} s_{-1-n} & yu & x^2 \end{vmatrix} = 0.$$

3. Somos-5. Hone [5] found general formulas for the elements of the Somos-5 sequence defined by the recurrence relation (3). They can be written as

$$s_{2n} = A_0 B_0^n \frac{\sigma(z_0 + 2nz)}{\sigma(z)^{(2n)^2}}, \qquad s_{2n+1} = A_1 B_1^n \frac{\sigma(z_0 + (2n+1)z)}{\sigma(z)^{(2n+1)^2}}. \tag{9}$$

A comparison with (4) shows that an arbitrary Somos-5 sequence can be viewed as a Somos-4 sequence whose odd-numbered elements are multiplied by some geometric progression. It follows from formulas (9) that the matrix $M_s^{(0)}$ has rank 4 in general position. However, each entry of $M_s^{(1)}$ is a product of even- and odd- numbered elements of the Somos-5 sequence. Therefore, the rank of the matrix $M_s^{(1)}$ for the Somos-5 sequence coincides with that of the matrix $M_s^{(1)}$ constructed from the Somos-4 sequence. Thus, the following assertion holds.

Theorem 4. *One has*

$$D_s^{(1)} \begin{pmatrix} m_1, m_2, m_3 \\ n_1, n_2, n_3 \end{pmatrix} = 0 \tag{10}$$

*for an arbitrary Somos-*5 *sequence* $\{s_l\}$ *and any integers* m_1, m_2, m_3 *and* n_1, n_2, n_3.

Remark 2. For the relation

$$D_s^{(0)} \begin{pmatrix} m_1, m_2, m_3 \\ n_1, n_2, n_3 \end{pmatrix} = 0 \tag{11}$$

to hold, one must additionally require that at least one of the two conditions $m_1 \equiv m_2 \equiv m_3 \pmod 2$ and $n_1 \equiv n_2 \equiv n_3 \pmod 2$ be satisfied. In this case, the proof of (11) also follows from (9).

Proof of Theorem 2. We simultaneously prove that the elements of the sequence $\{s_l\}$ belong to the ring $\mathbb{Z}[t, u, v, w, x, y, z]$ and that t divides s_l for $|l| \geqslant 3$. The proof will be carried out by induction, assuming that the desired assertions have already been proved for numbers with absolute value less than $|l|$. We will also assume that $|l| > 5$, because the assertion of the theorem for small l admits a straightforward verification.

For even $l = 2n$ ($|n| \geqslant 3$), we use the relation

$$D_s^{(1)} \begin{pmatrix} n, 0, -1 \\ n-1, 1, 0 \end{pmatrix} = \begin{vmatrix} s_{2n} y & s_{n+2} s_{n-1} & s_{n+1} s_n \\ s_n s_{1-n} & xv & ty \\ s_{n-1} s_{-n} & yu & tx \end{vmatrix} = 0,$$

which is a special case of Eq. (10). This relation can be reduced by equivalence transformations to the form

$$yt(x^2v-y^2u)s_{2n}=\begin{vmatrix} xvs_{n+1}s_n-tys_{n+2}s_{n-1} & s_ns_{1-n}\\ yus_{n+1}s_n-txs_{n+2}s_{n-1} & s_{n-1}s_{-n}\end{vmatrix}.$$

The divisibility of the resulting determinant by x^2v-y^2u can be justified in the same way as in the proof of Theorem 1. By the inductive assumption, all entries of a 2×2 matrix are divisible by t, which means that $s_{2n}y\in t\mathbb{Z}[t,u,v,w,x,y,z]$. In a similar way, considering the relation

$$D_s^{(1)}\begin{pmatrix} n-1,0,-1\\ n,1,0\end{pmatrix}=\begin{vmatrix} s_{2n}x & s_{n+1}s_{n-2} & s_ns_{n-1}\\ s_{n+1}s_{-n} & xv & ty\\ s_ns_{-1-n} & yu & tx\end{vmatrix}=0,$$

we find that $s_{2n}x\in t\mathbb{Z}[t,u,v,w,x,y,z]$. Therefore, $s_{2n}\in t\mathbb{Z}[t,u,v,w,x,y,z]$.

For odd $l=2n+1$ $(-4\leqslant n\leqslant 3)$, the induction step is justified based on the relations

$$D_s^{(1)}\begin{pmatrix} n-1,0,-1\\ n+1,1,0\end{pmatrix}=\begin{vmatrix} s_{2n+1}u & s_{n+1}s_{n-2} & s_ns_{n-1}\\ s_{n+2}s_{-n-1} & xv & ty\\ s_{n+1}s_{-n} & yu & tx\end{vmatrix}=0,$$

$$D_s^{(1)}\begin{pmatrix} n+1,0,-1\\ n-1,1,0\end{pmatrix}=\begin{vmatrix} s_{2n+1}v & s_{n+3}s_n & s_{n+2}s_{n+1}\\ s_ns_{1-n} & xv & ty\\ s_{n-1}s_{-n} & yu & tx\end{vmatrix}=0.$$

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