Math 491, Problem Set #8: Solutions

1. *Let a_n be the number of domino tilings of a 4-by-n rectangle, with $n \geq 0$ (we put $a_0 = 1$ by convention).*

   (a) *Prove that the sequence $a_0, a_1, \ldots$ satisfies a linear recurrence relation of order 16 or less.*

   For each vertical line through the tiling that divides one column of height 4 from the next, there are $2^4 = 16$ different ways in which one might see 0 to 4 horizontal dominos being pierced by the line. If we give each of these 16 patterns a symbol, then every domino tiling of a 4-by-n rectangle is represented by a string of $n + 1$ symbols from an alphabet of size 16, beginning and ending with the symbol that means "no horizontal dominos pierced by this line" (corresponding to the left end and right end of the rectangle); call this the first symbol. Conversely, such a codeword corresponds to exactly one tiling if certain forbidden adjancies do not occur, and otherwise corresponds to no tilings at all. Thus the number of (domino-)tilings of the 4-by-n rectangle is equal to the upper-left entry of the nth power of the 16-by-16 matrix whose i, jth entry tells whether the ith symbol can occur next to the jth symbol. By the Cayley-Hamilton theorem, the sequence consisting of these entries must satisfy the 16th-order linear recurrence relation given by the characteristic polynomial of the matrix.

   (b) *Prove that the sequence $a_0, a_1, \ldots$ satisfies a linear recurrence relation of order 8 or less.*

   To prove this claim, it's enough to show that half of the 16 symbols cannot occur in such a codeword (so that the 16-by-16 matrix can be replaced by an 8-by-8 matrix). To see this, note that half of the symbols indicate the presence of an odd number of horizontal dominos being pierced by a particular vertical line. That is, if you divide the rectangle in half along the vertical line and remove the horizontal dominos that the line pierces, one is left with two sub-regions, each with odd area. But then these two sub-regions cannot be tiled by dominos (since a domino has even

area). Hence only 8 of the 16 symbols can occur in codewords of the form $(1, \ldots, 1)$, and the claim is proved.

(c) *Prove that the sequence $a_0, a_1, \ldots$ satisfies a linear recurrence relation of order 6 or less.*

Now we show that only 6 of the original 16 symbols can actually participate in a codeword of the form $(1, \ldots, 1)$. Color the constituent squares of the rectangle black and white. If we have a tiling of the rectangle and we cut it along a vertical line (discarding the bisected pieces), the two remaining regions must be tilable by dominos; so each must contain as many black squares as it contains white squares. On the other hand, suppose we bisect the rectangle along the same line and *don't* discard the bisected pieces; then (since each column has an even number of squares) it's clear that each side must contain as many black squares as white. It follows that, among the bisected dominos, there must be as many with a black square on the left and a white square on the right as there are with a white square on the left and a black square on the right. This leaves only six possibilities: no bisected dominos; bisected dominos in rows 1 and 2; bisected dominos in rows 1 and 4; bisected dominos in rows 3 and 2; bisected dominos in rows 3 and 4; and bisected dominos in all four rows.

(d) *Prove that the sequence $a_0, a_1, \ldots$ does not satisfy a linear recurrence relation of order 2 or less.*

Suppose there were constants A, B, C, not all equal to zero, such that $Aa_n + Ba_{n-1} + Ca_{n-2} = 0$ for all n. Then we must have $Aa_0 + Ba_1 + Ca_2 = 0$, $Aa_1 + Ba_2 + Ca_3 = 0$, and $Aa_2 + Ba_3 + Ca_4 = 0$. Combining these three scalar equations into a single vector equation, we get $A(a_0, a_1, a_2) + B(a_1, a_2, a_3) + C(a_2, a_3, a_4) = (0, 0, 0)$. This tells us that the three vectors (a_0, a_1, a_2), (a_1, a_2, a_3), and (a_2, a_3, a_4) must be linearly dependent. But this would require that the matrix

$$\begin{matrix} a_0 & a_1 & a_2 \\ a_1 & a_2 & a_3 \\ a_2 & a_3 & a_2 \end{matrix}$$

have determinant zero. We can check that this determinant does

not vanish, since $a_0 = 1$, $a_1 = 1$, $a_2 = 5$, $a_3 = 11$, $a_4 = 36$, and

$$\begin{vmatrix} 1 & 1 & 5 \\ 1 & 5 & 11 \\ 5 & 11 & 36 \end{vmatrix} = 8 \neq 0.$$

Hence such constants A, B, C do not exist, which proves that the sequence of a_n's does not satisfy any linear recurrence of degree 2 or less.