Solutions to Midterm Exam

1. Use a parity argument to show that there does not exist a path from $(-3,-3)$ to $(3,3)$ (travelling by unit-steps in the $+x$, $-x$, $+y$, and $-y$ directions) that avoids the point $(0,0)$ and visits each of the other lattice points $\{(i,j) : -3 \leq i \leq 3,\ -3 \leq j \leq 3\}$ exactly once.

Solution: Color the point (i,j) black or white according to whether $i+j$ is even or odd. Every step takes you from a point of one color to a point of the other color. Since there are $7^2 - 1 = 48$ points to visit, you must take 47 steps. If you start from a black point, you must end on a white point. So you cannot start at $(-3,-3)$ and end at $(-3,-3)$ after taking 47 steps.

2. Use the pigeonhole principle to show that if you place 33 pebbles on a chessboard, there must be two pebbles that occupy the same square or are in adjacent squares (where "adjacent" means "horizontally or vertically adjacent").

Solution: Tile the chessboard by horizontal 1-by-2 rectangles, and let these 32 rectangles be the pigeonholes. If you place 33 pebbles on the board, two of them must lie in the same rectangle, and therefore be either in the same square or in adjacent squares.

3. How many odd numbers between 1,000,000 and 9,999,999 have distinct digits? Express your answer as a product of one-digit numbers.

Solution: There are 5 possibilities for the last digit (1, 3, 5, 7, or 9). There are then 8 possibilities for the first digit (the first digit cannot be equal to 0 or equal to the last digit). There are 8 remaining possibilities for the tens digit. There are 7 remaining possibilities for the hundreds digit. There are 6 possibilities for the thousands digit. There are 5 possibilities for the ten thousands digit. And there are 4 possibilities for the hundred thousands digit. So the total number of possibilities is $5 \times 8 \times 8 \times 7 \times 6 \times 5 \times 4$.

Note: If you do the digits from left to right, you run into the problem of not knowing how many possibilities there are for the last digit (it depends on how many of the digits 1,3,5,7,9 have already been seen). If you do the digits from right to left, you run into the problem of not knowing how many possibilities there are for the millions digit (it depends on whether you've already seen a 0). If you don't want the problem to split into cases and be expressed as a sum, you need to jump around among the digits.

4. How many 4-element subsets of $\{A, B, C, D, E, F, G, H\}$ contain at most one vowel? Express your answer as a number.

Solution: There are $\binom{8}{4}$ 4-element subsets, of which $\binom{6}{2}$ contain the A and the E (since, once you have chosen the A and the E, there are 6 consonants from which you must choose 2). So the answer is $\binom{8}{4} - \binom{6}{2} = 70 - 15 = 55$.

5. How many distinct 6-letter "words" can be formed from 3 A's, 2 B's, and a C? Express your answer as a number.

Solution: This is the same as asking for the number of permutations of the multiset $\{3 \cdot A, 2 \cdot B, 1 \cdot C\}$, of which there are $\binom{6}{3,2,1} = 6!/3!2!1 = 60$.

6. How many 20-combinations of the multiset $\{\infty \cdot a, \infty \cdot b, \infty \cdot c, \infty \cdot d\}$ are there such that a gets chosen at least once, b gets chosen at least twice, c gets chosen at least three times, and d gets chosen at least four times? Express your answer as a number.

Solution: Let w, x, y, z be the number of times a, b, c, and d get chosen, respectively, and let $w' = w - 1$, $x' = x - 2$, $y' = y - 3$, and $z' = z - 4$. Then the number of 20-combinations of the multiset satisfying the stated properties equals the number of quadruples (w', x', y', z') of non-negative integers satisfying $w' + x' + y' + z' = 10$. Using 1's and $*$'s, we see that this is the same as the number of permutations of $\{10 \cdot 1, 3 \cdot *\}$, or $\binom{13}{3}$.

Alternate solution: If we subtract 0, 1, 2, and 3 respectively (instead of 1, 2, 3, 4), we are counting the 14-combinations of the multiset in which each of the four elements gets chosen at least once. Following what Peter Wertz did last week, we can write a sequence of 14 1's, with 13 spaces separating them; we must put stars in 3 of these spaces, and we can do this in $\binom{13}{3}$ ways. $\binom{13}{3} = (13)(12)(11)/(3)(2)(1) = 286$.

7. Evaluate $\sum_{k=0}^{n} 2^k 3^{n-k} \binom{n}{k}$.

Solution: This is just the binomial theorem expansion of $(x + y)^n$ evaluated at $x = 3$ and $y = 2$, so the sum is 5^n.

8. Find the coefficient of t^3u^2v in the multinomial expansion of $(t-2u+4v)^6$. Express your answer as a number.

Solution: The coefficient is $\binom{6}{3,2,1}(1)^3(-2)^2(4)^1 = (6!/3!2!1!)(16) = 960$.