Math 475, Problem Set #9: Answers

A. *Brualdi, chapter 7, problem 28, parts (b), (c), and (e).*

(b) $1 - x + x^2 - \ldots + (-1)^n x^n + \ldots$ is a geometric series with initial term 1 and ratio $-x$, so the sum is $1/(1-(-x)) = 1/(1+x)$. (Note that this is the special case $c = -1$ of the result from part (a).)

(c) $\binom{\alpha}{0} + (-\binom{\alpha}{1})x + \binom{\alpha}{2}x^2 + \ldots + ((-1)^n\binom{\alpha}{n})x^n + \ldots = \binom{\alpha}{0} + \binom{\alpha}{1}(-x) + \binom{\alpha}{2}(-x)^2 + \ldots + \binom{\alpha}{n})(-x)^n + \ldots$, which we recognize as the generating function in the second Example on page 236, but with $x$ replaced by $-x$. So the sum is $(1-x)^\alpha$.

(e) $1 - \frac{1}{1!}x + \frac{1}{2!}x^2 - \ldots + (-1)^n\frac{1}{n!}x^n + \ldots = 1 + \frac{1}{1!}(-x) + \frac{1}{2!}(-x)^2 + \ldots + \frac{1}{n!}(-x)^n + \ldots = e^{-x}$.

B. *Brualdi, chapter 7, problem 29, parts (b), (d), and (e). (Note for part (b) that 0 is a multiple of 3.)*

(b) $(1 + x^3 + x^6 + \ldots)^4 = (1/(1-x^3))^4 = 1/(1-x^3)^4$.

(d) $(x + x^3 + x^{11})(x^2 + x^4 + x^5)(1 + x + x^2 + \ldots)(1 + x + x^2 + \ldots) = (x + x^3 + x^{11})(x^2 + x^4 + x^5)/(1-x)^2$.

(e) $(x^{10} + x^{11} + x^{12} + \ldots)^4 = (x^{10}/(1-x))^4 = x^{40}/(1-x)^4$.

C. *Brualdi, chapter 7, problem 30, part (d).*

Let $g(x) = \sum_{n=0}^{\infty} h_n x^n$. Summing the equation $h_n x^n = 8h_{n-1}x^n - 16h_{n-2}x^n$ for all $n \geq 2$, we get $g(x) - h_0 - h_1 x = 8(g(x) - h_0) - 16g(x)$, i.e., $g(x)+1 = 8x(g(x)+1) - 16x^2 g(x)$. This yields $(1-8x+16x^2)g(x) = -1+8x$, so $g(x) = (-1+8x)/(1-8x+16x^2)$. Expanding by partial fractions, we obtain $A/(1-4x) + B/(1-4x)^2$. Multiplying by $(1-4x)^2$, we get $-1+8x = A(1-4x)+B$. The values of $A$ and $B$ that make the LHS and RHS identically equal are $A = -2$ and $B = 1$. So $g(x) = -2/(1-4x) + 1/(1-4x)^2$. The coefficient of $x^n$ in $-2/(1-4x)$ is $(-2)(4)^n$ and the coefficient of $x^n$ in $1/(1-4x)^2$ is $(n+1)(4)^n$ (using formula (7.47) with $r = 4$ and $k = 2$). So $h_n = (-2)(4)^n + (n+1)(4)^n = (-2+n+1)4^n = (n-1)4^n$.

Note that this agrees with the answer from Assignment 8, problem E.

D. *Let $f_n$ be the Fibonacci sequence as defined at the bottom of page 211. In this problem you will use the method of section 7.5 to solve the nonhomogeneous recurrence relation $h_n = h_{n-1} + f_n$ with the initial condition $h_0 = 0$.*

(a) *Let $g(x) = \sum_{n=0}^{\infty} h_n x^n$, and show that $g(x) = \frac{x}{(1-x)(1-x-x^2)}$.*
Summing the equations $h_n x^n = h_{n-1}x^n + f_n x^n$ with $n$ going from 1 to infinity, and using the fact that $\sum_{n=0}^{\infty} f_n x^n = x/(1-x-x^2)$, we get $g(x) - h_0 = xg(x) + x/(1-x-x^2)$. Since $h_0 = 0$, this becomes $(1-x)g(x) = \frac{x}{1-x-x^2}$, so $g(x) = \frac{x}{(1-x)(1-x-x^2)}$.

(b) *By doing a partial fraction expansion of $g(x)$ of the form $g(x) = A/(1-x) + (B+Cx)/(1-x-x^2)$, derive a formula for $h_n$ in terms of Fibonacci numbers.*
We need to pick $A, B, C$ so that $A(1-x-x^2) + (B+Cx)(1-x)$ simplifies to $0 + 1x + 0x^2$; this means $A + B = 0$, $-A - B + C = 1$, and $-A - C = 0$, and we can easily solve this system of linear equations, obtaining $A = -1$, $B = 1$, and $C = 1$. So $g(x) = -1/(1-x) + (1+x)/(1-x-x^2) = -1/(1-x) + 1/(1-x-x^2) + x/(1-x-x^2)$. The coefficient of $x^n$ in $-1/(1-x)$ is $-1$, the coefficient of $x^n$ in $1/(1-x-x^2)$ is $f_{n+1}$, and the coefficient of $x^n$ in $x/(1-x-x^2)$ is $f_n$. Hence $h_n = -1 + f_{n+1} + f_n$.
(Remark: We saw in the chapter why the coefficient of $x^n$ in $x/(1-x-x^2)$ is $f_n$. To see why the coefficient of $x^n$ in $1/(1-x-x^2)$ is $f_{n+1}$, note that this is the same power series as $x/(1-x-x^2)$, but where every exponent is shifted down by 1. That is, we have $x/(1-x-x^{-}x^2) = x + x^2 + 2x^3 + 3x^4 + \ldots$ and $1/(1-x-x^{-}x^2) = 1 + x + 2x^2 + 3x^3 + \ldots$. So the coefficient of $x^n$ in $1/(1-x-x^2)$ is equal to the coefficient of $x^{n+1}$ in $x/(1-x-x^2)$, which is equal to $f_{n+1}$.)

(c) *Check your answer by comparing with formula (7.8) in Brualdi.*
Since $h_0 = 0$ and $h_n = h_{n-1} + f_n$, we have $h_n = f_1 + f_2 + \ldots + f_n$. Since $f_0 = 0$, this equals $f_0 + f_1 + f_2 + \ldots + f_n$, which is $s_n$. Brualdi showed that $s_n = -1 + f_{n+2}$. But this can be written as $-1 + f_{n+1} + f_n$, which agrees with what we saw in (b).

(The same method that we used here can also be applied to problem

A from assignment 8, removing the element of guesswork.)

E. *Brualdi, chapter 7, problem 32.*

First solution: Follow Brualdi's hint. Start with $1 + x + x^2 + \ldots = 1/(1-x)$. Multiply by $x$: $x + x^2 + x^3 + \ldots = x/(1-x)$. Differentiate: $1 + 2x + 3x^2 + \ldots = 1/(1-x)^2$. Multiply by $x$: $x + 2x^2 + 3x^3 + \ldots = x/(1-x)^2$. Differentiate: $1 + 4x + 9x^2 + \ldots = (1+x)/(1-x)^3$. Multiply by $x$: $x + 4x^2 + 9x^3 + \ldots = (x + x^2)/(1-x)^3$. Differentiate: $1 + 8x + 27x^2 + \ldots = (1 + 4x + x^2)/(1-x)^4$. Multiply by $x$ one last time: $0 + x + 8x^2 + 27x^3 + \ldots = (x + 4x^2 + x^3)/(1-x)^4$.

Second solution: The sequence $h_n = n^3$ satisfies the fourth-order recurrence relation $h_n - 4h_{n-1} + 6h_{n-2} - 4h_{n-3} + h_{n-4}$ with characteristic polynomial $r(x) = x^4 - 4x^3 + 6x^2 - 4x + 1 = (x-1)^4$. By Theorem 7.5.1 and the formula in the middle of page 233, the generating function $g(x)$ for the sequence $h_n$ must be of the form $p(x)/q(x)$ where $p(x)$ is a polynomial of degree $< 4$ where $q(x) = x^4 r(1/x) = 1-4x+6x^2-4x^3+x^4$. Write $p(x) = A+Bx+Cx^2+Dx^3$. We must have $p(x) = g(x)(1-4x+6x^2-4x^3+x^4)$, i.e., $A+Bx+Cx^2+Dx^3+0x^4+\ldots) = (h_0+h_1x+h_2x^2+h_3x^3+h_4x^4+\ldots)(1-4x+6x^2-4x^3+x^4)$. Equating terms, we get $A = h_0 = 0^3 = 0$, $B = h_1 - 4h_0 = (1)^3 - 4(0)^3 = 1$, $C = h_2-4h_1+6h_0 = (2)^3-4(1)^3+6(0)^3 = 4$, and $D = h_3-4h_2+6h_1-4h_0 = (3)^3 - 4(2)^3 + 6(1)^3 - 4(0)^3 = 1$. Hence $g(x) = (x + 4x^2 + x^3)/(1-x)^4$.

Remark: As a way of checking your answer, you can use the division discussed in class, to see if $x+4x^2+x^3$ divided by $1-4x+6x^2-4x^3+x^4$ really goes $0+1x+8x^2+27x^3+\ldots$. Or, better still, try multiplication: $(1-4x+6x^2-4x^3+x^4)(0+1x+8x^2+27x^3+64x^4+125x^5+\ldots) = (1\cdot 0)+(1\cdot 1-4\cdot 0)x+(1\cdot 8-4\cdot 1+6\cdot 0)x^2+(1\cdot 27-4\cdot 8+6\cdot 1-4\cdot 0)x^3+(1\cdot 64-4\cdot 27+6\cdot 8-4\cdot 1+1\cdot 0)x^4+(1\cdot 125-4\cdot 64+6\cdot 27-4\cdot 8+1\cdot 1)x^5+\ldots = 0 + 1x + 4x^2 + 1x^3 + 0x^4 + 0x^5 + \ldots$, which checks.