

Math 475, Problem Set #8: Answers

A. *Brualdi, problem 1, parts (a), (b), and (d).*

(a): As n goes from 1 to 6, the sum (call it h_n) takes on the values 1, 3, 8, 21, 55, and 144; we recognize these as Fibonacci numbers, so we conjecture that $h_n = f_{2n}$. To prove this by induction, we first note (base case) that for $n = 1$, $h_1 = f_1 = 1 = f_2 = f_{2n}$, and that for larger n , if we assume (induction hypothesis) that $h_n = f_{2n}$, then $h_{n+1} = h_n + f_{2n+1} = f_{2n} + f_{2n+1} = f_{2n+2} = f_{2(n+1)}$, as desired.

(b): As n goes from 0 to 6, the sum (call it h_n) takes on the values 0, 1, 4, 12, 33, 88, and 232; we recognize these as 1 less than Fibonacci numbers, so we conjecture that $h_n = f_{2n+1} - 1$. To prove this by induction, we first note (base case) that for $n = 0$, $h_0 = f_0 = 0 = 1 - 1 = f_1 - 1$, and that for larger n , if we assume (induction hypothesis) that $h_n = f_{2n+1} - 1$, then $h_{n+1} = h_n + f_{2(n+1)} = f_{2n+1} - 1 + f_{2n+2} = f_{2n+3} - 1$, as desired.

(d): As n goes from 0 to 6, the sum (call it h_n) takes on the values 0, 1, 2, 6, 15, 40, and 104; we recognize these as products of consecutive Fibonacci numbers, so we conjecture that $h_n = f_n f_{n+1}$. To prove this by induction, we first note (base case) that for $n = 0$, $h_0 = f_0^2 = 0 = (0)(1) = f_0 f_1$, and that for larger n , if we assume (induction hypothesis) that $h_n = f_n f_{n+1}$, then $h_{n+1} = h_n + (f_{n+1})^2 = f_n f_{n+1} + (f_{n+1})^2 = f_{n+1}(f_n + f_{n+1}) = f_{n+1} f_{n+2}$, as desired.

B. *Brualdi, problem 9.*

First solution: We imitate the example from the chapter, on page 223. Let $n \geq 2$. If the first square is blue, or the first square is white, then the coloring can be completed in h_{n-1} ways. If the first square is red, then the second square must be colored blue or white, and after that the coloring can be completed in h_{n-2} ways. Hence $h_n = h_{n-1} + h_{n-1} + 2h_{n-2} = 2h_{n-1} + 2h_{n-2}$. The characteristic equation is $x^2 - 2x - 2$, and the characteristic roots are $(2 \pm \sqrt{4 + 8})/2 = 1 \pm \sqrt{3}$, so the general formula for h_n is of the form $c_1(1 + \sqrt{3})^n + c_2(1 - \sqrt{3})^n$.

To solve for c_1 and c_2 , we use the initial conditions $h_0 = 1$ and $h_1 = 3$:

$$1 = c_1 + c_2,$$

$$3 = c_1(1 + \sqrt{3}) + c_2(1 - \sqrt{3}).$$

This system of linear equations has the solution $c_1 = \frac{1}{2} + \frac{1}{3}\sqrt{3}$, $c_2 = \frac{1}{2} - \frac{1}{3}\sqrt{3}$. So

$$h_n = \left(\frac{1}{2} + \frac{1}{3}\sqrt{3}\right)(1 + \sqrt{3})^n + \left(\frac{1}{2} - \frac{1}{3}\sqrt{3}\right)(1 - \sqrt{3})^n.$$

Second solution: We will show that this is the same problem as the Example on page 223 of the text, in a colorful disguise! Given a length- n string of a 's, b 's, and c 's in which no two a 's appear consecutively, we can create a correspond coloring of the 1-by- n chessboard in which (reading from left to right) each a corresponds to a red square, each b corresponds to a white square, and each c corresponds to a blue square. The result is a coloring in which no two squares that are colored red are adjacent. Furthermore, given any such coloring, we can represent each red square by an a , each white square by a b , and each blue square by a c ; the result is a string of a 's, b 's, and c 's in which no two a 's appear consecutively. Thus we have established a one-to-one correspondence between the words of length n discussed on page 223 and the colorings discussed in problem 7. It follows that the two sets are of the same size. That is, the number of permitted colorings of the 1-by- n chessboard equals the number of permitted words of length n , which (see the middle of page 224) is equal to $\frac{2+\sqrt{3}}{2\sqrt{3}}(1 + \sqrt{3})^n + \frac{-2+\sqrt{3}}{2\sqrt{3}}(1 - \sqrt{3})^n$.

- C. *Solve the recurrence relation $h_n = nh_{n-1} + n - 1$, ($n \geq 1$) with the initial value $h_0 = 0$. It is not enough to guess the pattern; you must prove it using induction.*

As n goes from 0 to 6, h_n takes on the values 0, 0, 1, 5, 23, 119, and 719; we recognize these as 1 less than the factorial numbers, so we conjecture that $h_n = n! - 1$. To prove this by induction, we first note (base case) that for $n = 0$, $h_0 = 0 = 1 - 1 = 0! - 1$, and that for larger n , if we assume (induction hypothesis) that $h_n = n! - 1$, then $h_{n+1} = (n+1)h_n + n = (n+1)(n! - 1) + n = (n+1)n! - (n+1) + n = (n+1)! - 1$, as desired.

Note that the recurrence is a linear recurrence but that it does not have constant coefficients; hence the methods of sections 7.2 and beyond do

not apply. That is why we had to resort to the method of section 7.1: guessing the answer, and then proving it by induction. If you guessed the right answer, but gave a specious justification saying things like “the characteristic polynomial is $x^2 - n$ ”, you will get fewer points than if you guessed the right answer and simply said “I don’t know how to prove this.”

(Note: even if you can’t guess the answer to a problem like this, you can still get partial credit for saying things that show some knowledge or insight, such as “The coefficient of h_{n-1} on the right-hand side isn’t a constant, so the methods we learned in section 7.3 and in class don’t apply.” However, if you write “the characteristic polynomial is $x^2 - nx + n - 1$,” then you’ve misunderstood an important point. This linear recurrence doesn’t *have* a characteristic polynomial. A characteristic polynomial must have constant coefficients; that is, it must be an equation in x alone. We can’t have n ’s floating around.)

D. *Brualdi, problem 14. (Use the method given in the book.)*

The characteristic equation $x^3 - x^2 - 9x + 9 = 0$ has roots 1, 3, and -3 , so the formula for h_n is of the form $h_n = c_1 + c_2 3^n + c_3 (-3)^n$. To solve for c_1 , c_2 , and c_3 , we use the initial conditions:

$$0 = h_0 = c_1 + c_2 + c_3,$$

$$1 = h_1 = c_1 + 3c_2 - 3c_3,$$

$$2 = h_2 = c_1 + 9c_2 + 9c_3.$$

If we subtract the first equation from each of the other two, we get

$$1 = 2c_2 - 4c_3,$$

$$2 = 8c_2 + 8c_3.$$

If we subtract 4 times the first new equation from the second new equation, we get

$$-2 = 24c_3.$$

So $c_3 = -1/12$. Plugging this back into either of the equations that involve only c_2 and c_3 , we can solve for c_2 , obtaining $c_2 = 1/3$. Plugging these two values back into the first equation, we get $c_1 = -1/4$. Hence

$h_n = (-1/4) + (1/3)3^n - (1/12)(-3)^n$. It's extremely easy to make mistakes in these calculations, so it's wise to check this by plugging in small values of n .

E. *Brualdi, problem 15. (Use the method given in the book.)*

The characteristic equation $x^2 - 8x + 16 = 0$ has 4 as a double root, so the formula for h_n is of the form $h_n = c_1 4^n + c_2 n 4^n$. To solve for c_1 and c_2 , we use the initial conditions:

$$-1 = h_0 = c_1$$

$$0 = h_1 = 4c_1 + 4c_2.$$

The first equation gives $c_1 = -1$, and plugging this into the second equation gives $c_2 = 1$. Hence $h_n = -4^n + n 4^n$.

F. *Brualdi, problem 26. (Use the method given in the book.)*

First, let me show the method you're *not* supposed to use (namely, the method I demonstrated in class), because in this problem I asked you to use Brualdi's method. The characteristic equation of the homogeneous recurrence is $x^2 - 6x + 9 = 0$. The sequence of nonhomogeneous correction terms $b_n = 2n$, satisfies the homogeneous recurrence $b_n - 2b_{n-1} + b_{n-2} = 0$; the characteristic equation of this recurrence is $x^2 - 2x + 1 = 0$. Therefore, by the theorem discussed in class, the sequence h_n has characteristic equation $(x^2 - 6x + 9)(x^2 - 2x + 1) = 0$, which has 1 as a double root and 3 as a double root. Therefore the formula for h_n is of the form $h_n = c_1 + c_2 n + c_3 3^n + c_4 n 3^n$. To solve for c_1 , c_2 , c_3 , and c_4 , we need h_0 , h_1 , h_2 , and h_3 . The first two are given to us by the initial conditions, we determine the other two by the recurrence: $h_2 = 6h_1 - 9h_0 + 4 = 6(0) - 9(1) + 4 = -5$, $h_3 = 6h_2 - 9h_1 + 6 = 6(-5) - 9(0) + 6 = -24$. Now we can set up a system of linear equations in the unknown coefficients c_1 , c_2 , c_3 , c_4 :

$$1 = h_0 = c_1 + c_3,$$

$$0 = h_1 = c_1 + c_2 + 3c_3 + 3c_4,$$

$$-5 = h_2 = c_1 + 2c_2 + 9c_3 + 18c_4,$$

$$-24 = h_3 = c_1 + 3c_2 + 27c_3 + 81c_4.$$

Solving, we get $c_1 = \frac{3}{2}$, $c_2 = \frac{1}{2}$, $c_3 = -\frac{1}{2}$, $c_4 = -\frac{1}{6}$. So $h_n = \frac{3}{2} + \frac{1}{2}n - \frac{1}{2}3^n - \frac{1}{6}n3^n$.

Secondly, let's do a hybrid solution to the problem, where our ability to guess a particular solution is informed by our understanding of the method discussed in class: Since the characteristic polynomial of the homogeneous recurrence is $x^2 - 6x + 9 = (x - 3)^2$, the general solution to the homogeneous recurrence is of the form $c_1 3^n + c_2 n 3^n$. Since we know (from the preceding solution) that there is a formula for h_n as a linear combination of the sequences 1^n , $n1^n$, 3^n , and $n3^n$, we guess that we can build a particular solution to the non-homogeneous recurrence by using the first two sequences as building blocks (since the last two are already appearing in the general solution to the homogeneous second-order recurrence). So, we guess that we can find a particular solution h_n to the original non-homogeneous recurrence, with h_n a linear combination of 1^n and $n1^n$, that is, with h_n a linear function of n . Write $h_n = a + bn$. Plugging this into the recurrence $h_n - 6h_{n-1} + 9h_{n-2} = 2n$, we get $(a + bn) - 6(a + b(n-1)) + 9(a + b(n-2)) = 2n$ or $(4a - 12b) + (4b)n = 2n$. Since this must be true for all n , we have $4a - 12b = 0$ and $4b = 2$, so that $b = 1/2$ and $a = 3/2$. So the general solution to the original non-homogeneous recurrence is given by $h_n = \frac{3}{2} + \frac{1}{2}n + c_1 3^n + c_2 n 3^n$. Now we can set up a system of linear equations in the unknown coefficients c_1, c_2 :

$$1 = h_0 = 3/2 + 0 + c_1 + 0,$$

$$0 = h_1 = 3/2 + 1/2 + 3c_1 + 3c_2.$$

Solving, we get $c_1 = -\frac{1}{2}$, $c_2 = -\frac{1}{6}$. So $h_n = \frac{3}{2} + \frac{1}{2}n - \frac{1}{2}3^n - \frac{1}{6}n3^n$.

Thirdly, here is the solution I'm sure Brualdi had in mind (and which I wanted you to rediscover in this assignment). Observe that the non-homogeneous part of the equation, $+2n$, is very similar to the non-homogeneous part of the equation given in the Example on page 230, $-4n$. Since the trick in that problem was to guess that there is a particular solution of the form $h_n = rn + s$, we should try the same thing here, writing $h_n = a + bn$. Now proceed as in the previous paragraph.