Math 475, Problem Set #7: Answers

A. *Let n be a positive integer ≥ 6. How many different ways are there of rolling n dice so that each of the numbers 1,2,...,6 occurs at least once? (Regard the dice as being distinguishable from one another.)*

Let A_i be the set of outcomes with i not appearing. $(i = 1, 2, 3, 4, 5, 6)$ $|A_i| = 5^n$ since there are five choices for each of the n dice, with i taken out. Similarly, $|A_i \cap A_j| = 4^n$, $|A_i \cap A_j \cap A_k| = 3^n$, $|A_i \cap A_j \cap A_k \cap A_l| = 2^n$, $|A_i \cap A_j \cap A_k \cap A_l \cap A_m| = 1$. So the number of ways of rolling n different dice so that each of the numbers $1, 2, 3, 4, 5, 6$ occurs at least once is $N(n) = 6^n - 6 \cdot 5^n + \binom{6}{2} \cdot 4^n - \binom{6}{3} \cdot 3^n + \binom{6}{4} \cdot 2^n - 6$.

Ask a check, note that $N(6) = 720 = 6!$. This makes sense, since the only way to achieve all six possible numbers in just six rolls is if all six numbers are distinct; that is, the six rolls form a permutation of $\{1, 2, 3, 4, 5, 6\}$, of which there are $6!$.

B. *Four married couples are seated around a circular table. How many arrangements are there if no wife sits next to her own husband? (Arrangements that differ only by rotation are to be regarded as identical.) We do not require men and women to alternate.*

Let A_i be the set of arrangements with couple i sitting together. $(i = 1, 2, 3, 4)$. $|A_i| = 2 \cdot 6!$: place husband i to the left or right of wife i and permute the remaining 6 people. $|A_i \cap A_j| = 2 \cdot 2 \cdot 5!$. This is because, if we think of couple i as a single entity and couple j as a single entity, and leave the remaining 4 as is, then the number of configurations of these 6 objects is $5!$, and the number of ways couple i can be ordered is 2, as is the number of ways that couple j can be ordered. Similarly, $|A_i \cap A_j \cap A_k| = 2 \cdot 2 \cdot 2 \cdot 4!$, $|A_i \cap A_j \cap A_k \cap A_l| = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 3!$. Hence, using inclusion-exclusion, the number of arrangements of four married couples so that no wife sits next to her own husband is $7! - 4 \cdot (2 \cdot 6!) + 6 \cdot (2^2 \cdot 5!) - 4 \cdot (2^3 \cdot 4!) + 2^4 \cdot 3!$. Note: If we require men and women to alternate around the table, this becomes a much harder problem!

C. *Given finite sets $A_1, A_2, ..., A_n$, let B be the set of all x that belong to at least* **two** *of the A_i's. (For instance, if $n = 3$ with $A_1 = \{T, H, I, S\}$,*

$A_2 = \{I, S\}$, and $A_3 = \{I, T\}$, then $B = \{S, I, T\}$.) By experimenting with small values of n (or by peeking ahead to part D), find a plausible general formula for the size of B in terms of the sizes $|A_i|$, $|A_i \cap A_j|$, $|A_i \cap A_j \cap A_k|$, etc. To get you started: when $n = 3$,

$$|B| = |A_1 \cap A_2| + |A_1 \cap A_3| + |A_2 \cap A_3| - 2|A_1 \cap A_2 \cap A_3|.$$

$$\begin{aligned} |B| \quad = \quad & \sum_{i<j} |A_i \cap A_j| \\ & - 2 \sum_{i<j<k} |A_i \cap A_j \cap A_k| \quad \text{(more)} \\ & + 3 \sum_{i<j<k<l} |A_i \cap A_j \cap A_k \cap A_l| \\ & \ldots \\ & + (-1)^n (n-1)\, |A_1 \cap A_2 \cap \ldots \cap A_n|. \end{aligned}$$

D. *Prove that for all integers $k \geq 2$,*

$$\binom{k}{2} - 2\binom{k}{3} + 3\binom{k}{4} - \ldots + (-1)^k (k-1)\binom{k}{k} = 1.$$

Binomial theorem proof: Starting from the identity

$$\binom{k}{0} + \binom{k}{1}x + \binom{k}{2}x^2 + \binom{k}{3}x^3 + \ldots = (1+x)^k,$$

we can subtract $\binom{k}{0} = 1$ from both sides, divide by x, and differentiate, obtaining

$$\binom{k}{2} + 2\binom{k}{3}x + 3\binom{k}{4}x^2 + \ldots = \frac{xk(1+x)^{k-1} - (1+x)^k + 1}{x^2}.$$

Substituting $x = -1$ now yields the desired result.

Nicer proof: Using the identity $i\binom{k}{i} = k\binom{k-1}{i-1}$, we get

$$\begin{aligned} 2\binom{k}{2} - 3\binom{k}{3} + 4\binom{k}{4} - \ldots &= k\binom{k-1}{1} - k\binom{k-1}{2} + \ldots \\ &= k\left[\binom{k-1}{1} - \binom{k-1}{2} + \ldots\right] \\ &= k\left[\binom{k-1}{0}\right] \\ &= k. \end{aligned}$$

If we subtract from this the known identity

$$\binom{k}{2} - \binom{k}{3} + \binom{k}{4} - \ldots = \binom{k}{1} - \binom{k}{0} = k-1,$$

we get the desired result.

Proof by Pascal's relation: We have $\binom{k}{2} - 2\binom{k}{3} + 3\binom{k}{4} - \ldots + (-1)^{k-1}(k-2)\binom{k}{k-1} + (-1)^k(k-1)\binom{k}{k} = (\binom{k-1}{1} + \binom{k-1}{2} - 2(\binom{k-1}{2} + \binom{k-1}{3}) + 3(\binom{k-1}{3} + \binom{k-1}{4}) - \ldots + (-1)^{k-1}(k-2)(\binom{k-1}{k-2} + \binom{k-1}{k-1}) + (-1)^k(k-1)(\binom{k-1}{k-1}) = \binom{k-1}{1} - \binom{k-1}{2} + \binom{k-1}{3} + \ldots + (-1)^k\binom{k-1}{k-1} = \binom{k-1}{0} = 1.$

Combinatorial proof (included for the sake of diehard fans of the combinatorial method of proof; all others beware!): From among k people we are to choose a committee of any (positive) number of people. The committee must be chaired by a person who is not the least competent person on the committee. Thus, the committee must have at least two members, and for all i from 2 to k, the number of ways of choosing an i-member committee and a chairperson is $(i-1)\binom{k}{i}$. Call such a committee-with-chairperson *positive* if it has an even number of members, and *negative* otherwise. We must show that the number of positive committees-with-chairperson is 1 greater than the number of negative committees-with-chairperson. We do this by pairing up even committees with odd committees, with one even committee left unpaired, as follows. Call the most competent of the k people Al, and the least competent person Zeke. Note that Zeke can never be chairman. If anyone other than Al is chairing the committee, then we can "switch" Al (that is, put him on the committee if he is off it, and take him off if he is on it), obtaining a committee of the opposite type. If on the other hand Al is chairing the committee, then we can switch Zeke (that is, put him on the committee if he is off it, and take him off if he is on it), obtaining a committee of the opposite type, *unless* the committee consists of just Al and Zeke (for in this case, removing Zeke would give rise to a 1-person committee, chaired by its only and hence least competent member). This two-person committee consisting of Al and Zeke is a positive committee. Every other committee is paired up with a committee of the opposite type. This proves the result.

E. *Prove your formula from part C. (Hint: You will probably find the result of part D useful.)*

If an element x belongs to 0 or 1 of the sets A_i, then it clearly makes no contribution to the right hand side. On the other hand, suppose x belongs to k of the sets A_i with $2 \leq k \leq n$. Then it contributes $+1$ via each of $\binom{k}{2}$ terms of the form $|A_i \cap A_j|$, -2 via each of $\binom{k}{3}$ terms of the form $|A_i \cap A_j \cap A_k|$, and so on, giving a total contribution of

$$\binom{k}{2} - 2\binom{k}{3} + 3\binom{k}{4} - \ldots + (-1)^k(k-1)\binom{k}{k} = 1,$$

as required.

F. *Use the formula from part C to obtain a solution to problem 15c in Brualdi.*

Let A_i denote the set of ways for the 7 gentlemen's hat to be returned such that the ith gentleman gets his own hat back. Then the number of ways for two or more gentlemen to get their hats back is exactly $|B|$. Applying the formula, we get

$$|B| = \binom{7}{2}5! - 2\binom{7}{3}4! + 3\binom{7}{4}3! - 4\binom{7}{5}2! + 5\binom{7}{6}1! - 6\binom{7}{7}0!.$$

It is easy to check that both this and the answer given in the back of Brualdi are numerically equal to 1331.