Math 475, Problem Set #6: Solutions

A. (a) *For each point* (a, b) *with* a, b *non-negative integers satisfying* $a+b \leq 8$*, count the paths from (0,0) to* (a, b) *where the legal steps from* (i, j) *are to* $(i+2, j)$*,* $(i, j+2)$*, and* $(i+1, j+1)$*.*

I'll do this in the first quadrant using i and j as x- and y-coordinates (though you could use the i and j as row-index and column-index; you'd get different pictures but the same numbers).

1								
0	4							
1	0	10						
0	3	0	16					
1	0	6	0	19				
0	2	0	7	0	16			
1	0	3	0	6	0	10		
0	1	0	2	0	3	0	4	
1	0	1	0	1	0	1	0	1

(b) *Compute the coefficients of* $(x^2+xy+y^2)^n$ *for* $n = 0, 1, 2, 3, 4$*.*

$(x^2+xy+y^2)^0 = 1$, $(x^2+xy+y^2)^1 = x^2+xy+y^2$, $(x^2+xy+y^2)^2 = x^4+2x^3y+3x^2y^2+2xy^3+y^4$, $(x^2+xy+y^2)^3 = x^6+3x^5y+6x^4y^2+7x^3y^3+6x^2y^4+3xy^5+y^6$, and $(x^2+xy+y^2)^4 = x^8+4x^7y+10x^6y^2+16x^5y^3+19x^4y^4+16x^3y^5+10x^2y^6+4xy^7+y^8$.

(c) *Based on parts (a) and (b), formulate a precise conjecture of the form "for all non-negative integers* a *and* b*, the number of paths from* $(0,0)$ *to* (a,b) *is equal to the coefficient of* $\ldots$ *in the polynomial* $\ldots$*".*

The number of paths from $(0,0)$ to (a,b) is equal to the coefficient of $x^a y^b$ in $(x^2+xy+y^2)^{(a+b)/2}$. (This assumes that $a+b$ is even; if $a+b$ is odd, the number of paths from $(0,0)$ to (a,b) is zero.) You can think of the three terms of x^2+xy+y^2 as representing the three possible steps you're allowed to take: $x^2 = x^2y^0$ corresponds to moving 2 steps to the right, $xy = x^1y^1$ corresponds to moving 1 step to the right and 1 step up, and $y^2 = x^0y^2$ corresponds to moving 2 steps upward. The terms in the expansion of $(x^2+xy+y^2)^m$ correspond to all the places (a,b)

you can get to using m steps of the allowed kinds, and the coefficient of $x^a y^b$ in this expansion is the number of ways to get there.

B. *Chapter 5, problem 12.*

Algebraic proof: Rewriting the expression as

$$\binom{n}{0}\binom{n}{n} - \binom{n}{1}\binom{n}{n-1} + \binom{n}{2}\binom{n}{n-2} - \dots + (-1)^n\binom{n}{n}\binom{n}{0},$$

we see that it is the coefficient of x^n in the product of

$$\binom{n}{0}x^n - \binom{n}{1}x^{n-1} + \binom{n}{2}x^{n-2} - \dots + (-1)^n\binom{n}{n}$$

and

$$\binom{n}{0}x^n + \binom{n}{1}x^{n-1} + \binom{n}{2}x^{n-2} + \dots + \binom{n}{n}.$$

But by applying the binomial theorem to the two factors, we see that this polynomial is equal to $(x-1)^n(x+1)^n = (x^2-1)^n$. All the terms in this sum are of even degree, so if n is odd, the coefficient of x^n in the polynomial vanishes. On the other hand, if n is even (say $n = 2k$), then, setting $y = x^2$, we see that the coefficient of x^n in $(x^2-1)^n$ is the same as the coefficient of $y^{n/2}$ in $(y-1)^n$, which is $(-1)^{n/2}\binom{n}{n/2}$.

(Remark: Even if you didn't find this, you could have still gotten partial credit if you noticed that, when n is odd, the terms of the alternating sum of the squares of the binomial coefficients cancel in pairs: $\binom{n}{0}^2$ cancels $-\binom{n}{n}$, $-\binom{n}{1}^2$ cancels $+\binom{n}{n-1}$, etc.)

Combinatorial proof: Imagine n men and n women, from whom we wish to choose n individuals to form a committee. Call a committee **even** if it contains an even number of women, and **odd** otherwise. Then

$$\binom{n}{0}\binom{n}{n} + \binom{n}{2}\binom{n}{n-2} + \dots$$

is the number of even committees, and

$$\binom{n}{1}\binom{n}{n-1} + \binom{n}{3}\binom{n}{n-3} + \dots$$

is the number of odd committees. We wish to show that the number of even commitees minus the number of odd commitees is 0 when n is odd and $(-1)^{n/2}\binom{n}{n/2}$ when n is even. We will do this by pairing up even committees with odd committees in such a way that when n is odd, there are no unpaired committees, while if n is even, there are exactly $\binom{n}{n/2}$ unpaired committees, all of which are even if $n/2$ is even and odd if $n/2$ is odd. To accomplish this, first marry off the men and women, and number the resulting couples from 1 to n. If a committee consists completely of married couples, we don't pair it with another committee; otherwise, we pair it with the committee obtained by replacing the lowest-numbered committtee member whose spouse is not on the committee by that committee-member's spouse. (Here "lowest-numbered" could mean "with the lowest social security number", or anything else that lets us break ties.) This pairing clearly pairs odd committees with even committees. If n is odd, there are no unpaired committees. If n is even, then there are $\binom{n}{n/2}$ unpaired committees, each of which is either even or odd according to the parity of $n/2$.

C. *Solve Brualdi, Chapter 5, problem 18 in two different ways: once using problem 16 as a model, and once using problem 17 as a model.*

First method: Integrating $\sum_{k=0}^{n} x^k\binom{n}{k} = (1+x)^n$ from $x=0$, we get $\sum_{k=0}^{n}\frac{1}{k+1}x^{k+1}\binom{n}{k} = \frac{1}{n+1}((1+x)^{n+1}-1)$, and setting $x=-1$, we get $\sum_{k=0}^{n}(-1)^{k+1}\frac{1}{k+1}\binom{n}{k} = \frac{1}{n+1}(0^{n+1}-1) = \frac{-1}{n+1}$. Multiplying by -1, we get $\sum_{k=0}^{n}(-1)^k\frac{1}{k+1}\binom{n}{k} = \frac{1}{n+1}$.

Second method:

$$\begin{aligned}
& \binom{n}{0} - \frac{1}{2}\binom{n}{1} + \frac{1}{3}\binom{n}{2} - \ldots + (-1)^n\frac{1}{n+1}\binom{n}{n} \\
= & \frac{1}{n+1}\left[\binom{n+1}{1} - \binom{n+1}{2} + \binom{n+1}{3} - \ldots + (-1)^n\binom{n+1}{n+1}\right] \\
= & \frac{1}{n+1}\binom{n+1}{0} \\
= & \frac{1}{n+1}\,.
\end{aligned}$$

D. *What is the coefficient of $x_1^3x_2^3x_3x_4^2$ in the expansion of $(x_1 - x_2 + 2x_3 - 2x_4)^9$?*

The relevant term of the multinomial theorem is equal to the product of

$$\binom{9}{3\ 3\ 1\ 2} = \frac{9!}{3!3!1!2!} = 5040$$

and

$$(x_1)^3(-x_2)^3(2x_3)^1(-2x_4)^2 = -8x_1^2x_2^3x_3^1x_4^2;$$

the total coefficient is thus $(5040) \times (-8) = -40320$.

E. *Brualdi, Chapter 5, problem 46. Retain all terms that are greater than 10^{-3}; discard the rest.*

Let $x = 2$ and $y = 8$. Since $0 \le |2| \le |8|$, Theorem 5.6.1 applies: $(2+8)^{1/3} = \binom{1/3}{0}2^0 8^{1/3} + \binom{1/3}{1}2^1 8^{1/3-1} + \binom{1/3}{2}2^2 8^{1/3-2} + \binom{1/3}{3}2^3 8^{1/3-3} + \binom{1/3}{4}2^4 8^{1/3-4} + \ldots = (1)(2)+(1/3)(2^{-1})+(\frac{(1/3)(-2/3)}{2!})(2^{-3})+(\frac{(1/3)(-2/3)(-5/3)}{3!})(2^{-5})+(\frac{(1/3)(-2/3)(-5/3)(-8/3)}{4!})(2^{-7})+\ldots = 2+1/6-1/72+5/2592-5/15552+\ldots$. These terms decrease in absolute value and alternate in sign (leaving aside the first two terms), and the fifth term is less than 10^{-3}, so we may approximate the infinite sum to within 10^{-3} by taking just the first four terms: $2 + 1/6 - 1/72 + 5/2592 = 5585/2592 = 2.1547\ldots$. (The true value is $2.1544\ldots$.)

F. *Fix positive integers $n, k \ge 3$. Consider a convex n-gon with vertices labelled 1 through n. Call a convex k-gon, whose vertices are a subset of the vertices of the n-gon, an internal k-gon if all of its sides are diagonals of the n-gon.*

(a) *How many internal k-gons are there containing the vertex labelled 1?*

The internal k-gons containing vertex 1 are in 1-1 correspondence with the k-tuples $(x_1, x_2, ..., x_k)$ satisfying $x_1+x_2+...+x_k = n-k$, where each x_i is a positive integer. (Think of x_i as the number of vertices of the n-gon that lie between successive vertices of the k-gon.) Letting $y_i = x_i-1$, we see that these in turn correspond to k-tuples $(y_1, y_2, ..., x_k)$ satisfying $y_1+y_2+...+y_k = n-2k$ where each y_i is a non-negative integer. Applying one-and-stars in the usual way we see that the number of such k-tuples is $\binom{(n-2k)+(k-1)}{k-1} = \binom{n-k-1}{k-1}$.

(b) *How many internal k-gons are there all together? (Hint: What do you know ahead of time about the ratio between the answer to (a) and the answer to (b)?)*

Claim (a) is true about any particular vertex, and not just the vertex labelled 1. So if we multiply $\binom{n-k-1}{k-1}$ by n (the number of vertices) and divide by k (to take into account the fact that each k-gon has k different vertices), we get the correct answer, namely

$$\frac{n}{k}\binom{n-k-1}{k-1} = \frac{n(n-k-1)!}{k!(n-2k)!}.$$

Alternative analysis: Let's count, in two different ways, the number of ways to draw an internal k-gon and pick a vertex of that k-gon and color it blue. On the one hand, the answer is N_b times k, where N_b is the answer to problem (b), since there are N_b possible internal k-gons, each of which has k vertices that are available to be colored blue. On the other hand, you could choose the blue vertex first. There are n vertices of the original n-gon that could be colored blue. For each choice of the blue vertex, there are N_a internal k-gons that contain that vertex, where N_a is the answer to part (a). (We proved this in the case where the special vertex is vertex 1, but there's nothing special about the vertex that we happened to call vertex 1; the answer is N_a for any particular vertex.) Hence $N_b k = nN_a$, from which it follows that $N_b = (n/k)N_a$.