Math 475, Problem Set #2
(due 2/5/04)

A. *Section 2.4, problem 5.*

Divide the integers from 1 to $3n$ into n triples: 1 through 3, 4 through 6, 7 through 9, etc. If $n+1$ integers between 1 and $3n$ are chosen, two of them must lie in the same triple; but these two must then differ by at most 2.

B. *Section 2.4, problem 9. Omit the last sentence.*

It's enough to solve the simpler-seeming problem in which the words "(with no common person)" are removed. For, if we can find two sets A and B with the desired property, then so do the smaller sets $A' = A - (A \cap B)$ and $B' = B - (A \cap B)$, which have no person in common.

So, for each of the $2^{10} - 1 = 1023$ non-empty subsets of the 10 people present, consider the sum of the ages of the people in that set. The sum must be between 1 and 600 inclusive (since each individual person's age is between 1 and 60). Thus there are 600 different age-sums that can occur. Since there are 1023 non-empty subsets and only 600 age-sums, two of the sub-sets must have the same age-sum. By the remark made in the preceding paragraph, means that two groups of people in the room with no person in common must have the same age-sum.

Note: Some of the students included the empty set along with the 1023 non-empty sets in their use of the pigeonhole principle. This is okay too; it'll still be the case that the two sets A and B with the same age-sum (whose existence is guaranteed by the pigeonhole principle) are distinct, so it'll still be the case that the disjoint sets A' and B' are distinct, From this one can show that the sets A' and B' are both non-empty. (If one of them were empty, its age-sum would be zero; but then the age-sum of the other set would have to be zero, so the other set would have to be empty too, contradicting the fact that the two sets are distinct from one another.)

If the above seems at all confusing, make sure you understand the difference between saying that two sets are distinct and saying that two sets are disjoint.

C. *Section 2.4, problem 14.*

After $11 + 11 + 11 + 11 + 1 = 45$ minutes, I am assured of having picked at least a dozen pieces of fruit of the same kind. For, imagine that I put the fruit I pick into bins, according to type. If the apple-bin contains fewer than a dozen apples, and the banana-bin contains fewer than a dozen bananas, and the orange-bin contains fewer than a dozen oranges, and the pear-bin contains fewer than a dozen pears, then all the bins taken together contain at most $11 + 11 + 11 + 11 = 44$ pieces of fruit. But after 45 minutes, I have picked 45 pieces of fruit.

D. (a) *Find a sequence of 12 distinct numbers that contains no increasing subsequence of length 4 or decreasing subsequence of length 5.*

Example: 10,11,12,7,8,9,4,5,6,1,2,3. Group the terms into 4 blocks of length 3. Every increasing subsequence must lie within a single block, so it cannot have length 4. Every decreasing subsequence must have all its terms in different blocks, so it cannot have length 5.

(b) *Show that every sequence of 13 distinct numbers must contain either an increasing subsequence of length 4 or a decreasing subsequence of length 5.*

Suppose $a_1, a_2, ..., a_{13}$ are distinct. Let I_k (resp. D_k) be the length of the longest increasing (resp. decreasing) subsequence having a_k as its first term. If I_k takes only the values $1, 2, 3$ and D_k takes only the values $1, 2, 3, 4$ then the pair (I_k, D_k) takes only $3 \times 4 = 12$ values. Hence two of the thirteen pairs (I_k, D_k) must be the same. But, as was shown in class, this is impossible: for if $a_i < a_j$ then $I_i > I_j$, while if $a_i > a_j$ then $D_i > D_j$.

(c) *Formulate and prove a generalization of the Erdös-Szekeres theorem (Brualdi's "Application 9") in which the length of the desired increasing subsequence is $r + 1$ and the length of the desired decreasing subsequence is $s + 1$. Your theorem should contain both the Erdös-Szekeres theorem and part (b) of this problem as special cases.*

Every sequence of $rs + 1$ distinct terms contains an increasing subsequence of length $r + 1$ or a decreasing subsequence of length $s + 1$. Proof: Suppose $a_1, a_2, ..., a_{rs+1}$ are distinct. Let I_k (resp.

D_k) be the length of the longest increasing (resp. decreasing) subsequence having a_k as its first term. If I_k takes only the values $1, ..., r$ and D_k takes only the values $1, ..., s$ then the pair (I_k, D_k) takes only rs values. Hence two of the rs pairs (I_k, D_k) must be the same. But this is impossible. Hence, either there exists k with $I_k > r$ or there exists k with $D_k > s$.

Note: These solutions to (b) and (c) are based on the proof of the Erdös-Szekeres theorem that I did in class; one could also base solutions to these homework problems on the proof given in Brualdi's book.

E. *Given 11 real numbers represented as infinite decimals, show that two of them must agree at infinitely many decimal places.*

There are only 10 decimal digits, so for each positive integer n, some pair of the 11 numbers must have the same digit in the nth position after the decimal point. Let $A_{i,j}$ (with $1 \leq i < j \leq 11$) be the set of n's such that the ith and jth numbers in the list agree in the nth position. We have already seen that every positive integer n is in one of the $A_{i,j}$'s; since the set of positive numbers is infinite, and since there are only finitely many (specifically, 55) sets $A_{i,j}$ under consideration, one of these sets must be infinite. For that pair i, j, we see that the ith and jth numbers agree infinitely often.

(We'll learn in the next chapter why there are exactly 55 ways to choose integers i, j with $1 \leq i < j \leq 11$.)

Alternate proof: Suppose all the sets $A_{i,j}$ are finite. Then their union must be finite also, since there are only finitely many of these sets. But the union of the $A_{i,j}$ is the infinite set $\{1, 2, 3, \ldots\}$. This contradiction shows that one of the sets $A_{i,j}$ is infinite.

(Note that the sets $A_{i,j}$ are not disjoint; so this problem is a bit different from the standard set-up for the pigeonhole principle, in which each pigeon gets assigned to one and only one pigeonhole.)