Math 475, Problem Set #11: Answers

A. *Chapter 8, problem 2.*

We can put these arrays into one-to-one correspondence with acceptable sequences of +1's and −1's. Given such an array, define a_k (for all k between 1 and $2n$) to be +1 if k appears in the first row and −1 if k appears in the second row. E.g., the array

$$\begin{bmatrix} 1 & 3 & 4 & 6 \\ 2 & 5 & 7 & 8 \end{bmatrix}$$

corresponds to the sequence $+1, -1, +1, +1, -1, +1, -1, -1$. This sequence must contain n +1's and n −1's, since the array contains n entries in its first row and n entries in its second row. Furthermore, the sequence must be acceptable (in the sense defined on page 268). For, suppose the partial sum $a_1 + \ldots + a_k$ were negative. Let s (respectively, t) be the number of positive (respectively, negative) terms in the partial sequence $a_1, \ldots, a_k$. Then the tth column of the array contains a k in its second row and an entry larger than k in its first row, contradicting the condition stated in the problem. Hence the sequence $a_1, \ldots, a_{2n}$ is acceptable. Conversely, given any acceptable sequence of n +1's and n −1's, we can create a valid array by listing in the first row (in increasing order) all the k's for which $a_k = +1$ and listing in the second row (in increasing order) all the k's for which $a_k = -1$.

So the number of arrays satisfying the stated conditions equals the number of acceptable sequences, which is C_n. ($\frac{1}{n+1}\binom{2n}{n}$ and $\frac{(2n!)}{n!(n+1)!}$ are other acceptable answers.)

B. *Find (and prove) a formula for the number of integer sequences $a_1, a_2, \ldots, a_n$ with $1 \leq a_1 \leq a_2 \leq \ldots \leq a_n \leq n$ and $a_k \geq k$ for all k.*

Guessing the formula: We systematically list all the possibilities, for some small values of n. E.g., when $n = 1$, the only possibility is $a_1 = 1$. When $n = 2$, there are two possibilities: $(a_1, a_2) = (1, 2)$ and $(a_1, a_2) = (2, 2)$. For $n > 2$, we need to be systematic. Here's an example of how to be systematic when $n = 3$:

a_1 can be 1, 2, or 3.

If $a_1 = 1$, a_2 can be 2 or 3.

If $a_1 = 1$ and $a_2 = 2$, then a_3 can only be 3.

If $a_1 = 1$ and $a_2 = 3$, then a_3 can only be 3.

If $a_1 = 2$, a_2 can be 2 or 3.

If $a_1 = 2$ and $a_2 = 2$, then a_3 can only be 3.

If $a_1 = 2$ and $a_2 = 3$, then a_3 can only be 3.

If $a_1 = 3$, a_2 can only be 3.

If $a_1 = 3$ and $a_2 = 3$, then a_3 can only be 3.

So the possibilities for (a_1, a_2, a_3) are (1,2,3), (1,3,3), (2,2,3), (2,3,3), and (3,3,3) (five possibilities all told). Similarly, for $n = 4$, one can check that the possibilities for (a_1, a_2, a_3, a_4) are (1,2,3,4), (1,2,4,4), (1,3,3,4), (1,3,3,4), (1,4,4,4), (2,2,3,4), (2,2,4,4), (2,3,3,4), (2,3,4,4), (2,4,4,4), (3,3,3,4), (3,3,4,4), (3,4,4,4), and (4,4,4,4) (fourteen possibilities all told). So, the number of possibilities goes 1, 2, 5, 14, ... as n goes from 1 to infinity, and it's natural to conjecture that the answer is C_n.

Proving the formula: We can put these sequences into one-to-one correspondence with the paths discussed in the Example on page 271 that stay above diagonal. For example, in the case $n = 4$ (shown on page 271), consider the path P that goes north, north, east, north, east, north, east, east. If we look under this path (more precisely, if we look in the region bounded between the path P and the "reference path" P_0 that goes east, east, east, east, north, north, north, north), we see a stack of 2 squares, and to the right of that, a stack of 3 squares, and to the right of that, a stack of 4 squares, and to the right of that, a stack of 4 squares. This gives us the sequence 2, 3, 4, 4.

More generally, if we have a path P consisting of n eastward steps and n northward steps, we can look at the height of the kth eastward step in the picture (which is equal to the number of northward steps that precede the kth eastward steps), and call this a_k. Then we have $a_1 \leq a_2 \leq \ldots \leq a_n$, and moreover, the fact that the path P never crosses below the diagonal implies that $a_k \geq k$ for $k = 1, 2, \ldots, n$.

Conversely, every sequence $1 \leq a_1 \leq a_2 \leq \ldots \leq a_n \leq n$ with $a_1 \geq 1$, $a_2 \geq 2$, $\ldots$, $a_n \geq n$ gives a path P that goes from Home to Office without ever crossing below the diagonal line that joins Home to Office.

Therefore, the number of sequences $a_1, \ldots, a_n$ satisfying the stated conditions equals the number of paths from Home to Office that never go below the diagonal, which we know is equal to C_n.

C. *Repeat the Example from the middle of page 270, but this time assume that the cash register starts with a single 50 cent piece (rather than starting empty). We still assume that there are $2n$ people in line to get into the theatre, that admission costs 50 cents, that n of the people in line have a 50 cent piece and n of them have a 1 dollar bill. In how many ways can the people line up so that whenever a person with a 1 dollar bill buys a ticket, the box office has a 50 cent piece in order to make change?*

For $1 \leq i \leq 2n$, let a_i be $+1$ if the ith person in line has a 50 cent piece and -1 otherwise, so that $\sum_{i=1}^{2n} a_i = 0$. Then the cashier will always be able to make change right away provided that no partial sum of the a_i sequence is less than -1. Call a sequence of n $+1$'s and n -1's *inadmissable* if one of its partial sums is less than -1, and *admissable* otherwise. We can show that the number of inadmissable sequences is $\binom{2n}{n-2}$. For, suppose the sequence $a_1, \ldots, a_{2n}$ is inadmissable. Then there must exist a smallest k such that the partial sum $a_1 + a_2 + \ldots + a_k$ is less than -1. Because k is the smallest such value, we have $a_1 + a_2 + \ldots + a_{k-1} = -1$ and $a_k = -1$. We now reverse the signs of each of the first k terms. The resulting sequence $a'_1, a'_2, \ldots, a'_{2n}$ is a sequence of $(n+2)$ $+1$'s and $(n-2)$ -1's. The process is reversible, just as in the Example. So there are as many inadmissable sequences as there sequences of $(n+2)$ $+1$'s and $(n-2)$ -1's. That is, the number of inadmissable sequences is $\binom{2n}{n+2}$. Since the total number of sequences of n $+1$'s and n -1's is $\binom{2n}{n}$, the number of admissable sequences is $\binom{2n}{n} - \binom{2n}{n+2}$.

Alternative solution: Define "admissable" sequences as in the previous paragraph. If we are given an admissable sequence, so that all the partial sums are ≥ -1, sticking an extra $+1$ at the front and an extra

-1 at the end gives rise to an acceptable sequence (since all the partial sums are now ≥ 0, and the sum of all the terms is still 0). Conversely, every acceptable sequence must start with a $+1$ and end with a -1, and removing these two terms gives rise to an admissable sequence. Thus there is a one-to-one correspondence between the *admissable* sequences containing n $+1$'s and n -1's and the *acceptable* sequences containing $(n+1)$ $+1$'s and $(n+1)$ -1's. Thus the number of admissable sequences is C_{n+1}, the $n+1$st Catalan number. (It can be checked that $\binom{2n}{n} - \binom{2n}{n+2} = \frac{1}{n+2}\binom{2n+2}{n+1}$, so the two methods have given the same answer.)

D. *Chapter 8, problem 7. Express both* h_n *and* $\sum_{k=0}^{n} h_k$ *as polynomials in* n *in the ordinary way.*

The difference table for h_n is

$$\begin{array}{ccccccc} 1 & & -1 & & 3 & & 10 \\ & -2 & & 4 & & 7 & \\ & & 6 & & 3 & & \\ & & & -3 & & & \end{array}$$

So we have $h_n = 1\binom{n}{0} - 2\binom{n}{1} + 6\binom{n}{2} - 3\binom{n}{3} = -\frac{1}{2}n^3 + \frac{9}{2}n^2 - 6n + 1$ and $\sum_{k=0}^{n} h_k = 1\binom{n+1}{1} - 2\binom{n+1}{2} + 6\binom{n+1}{3} - 3\binom{n+1}{4} = -\frac{1}{8}n^4 + \frac{5}{4}n^3 - \frac{7}{8}n^2 - \frac{5}{4}n + 1$.

E. *Chapter 8, problem 8. Express* $\sum_{k=1}^{n} k^5$ *as a polynomial in* n *in the ordinary way.*

Note that $\sum_{k=1}^{n} k^5 = \sum_{k=0}^{n} k^5$.

The difference table for fifth powers is

$$\begin{array}{ccccccccccc} 0 & & 1 & & 32 & & 243 & & 1024 & & 3125 \\ & 1 & & 31 & & 211 & & 781 & & 2101 & \\ & & 30 & & 180 & & 570 & & 1320 & & \\ & & & 150 & & 390 & & 750 & & & \\ & & & & 240 & & 360 & & & & \\ & & & & & 120 & & & & & \end{array}$$

so the polynomial for adding fifth powers is $0\binom{n+1}{1} + 1\binom{n+1}{2} + 30\binom{n+1}{3} + 150\binom{n+1}{4} + 240\binom{n+1}{5} + 120\binom{n+1}{6} = \frac{1}{6}n^6 + \frac{1}{2}n^5 + \frac{5}{12}n^4 - \frac{1}{12}n^2$.