Math 475, Problem Set #10: Answers

A. *Consider the sequence 1, 2, 8, 40, 224, 1344, 8448, 54912, ... defined by the initial condition* $a_1 = 1$ *and the recurrence relation* $a_n = 2(a_1 a_{n-1} + a_2 a_{n-2} + \ldots + a_{n-1} a_1)$ *(valid for all* $n \geq 2$*). Find (and prove) a general formula for* a_n*.*

First solution: We imitate the analysis on pages 252–253. Let $g(x) = h_1 + h_2 x^2 + h_3 x^3 + \ldots$ be the generating function for the (new) sequence $h_1, h_2, h_3, \ldots$. Multiplying $g(x)$ by itself, and using the recurrence relation (7.52) and the fact that $h_1 = 1$, we get

$$\begin{aligned} 2(g(x))^2 &= 2h_1h_1x^2 + 2(h_1h_2 + h_2h_1)x^3 + 2(h_1h_3 + h_2h_2 + h_3h_1)x^4 + \ldots \\ &= h_2x^2 + h_3x^3 + h_4x^4 + \ldots \\ &= g(x) - h_1x \\ &= g(x) - x. \end{aligned}$$

Thus $g(x)$ satisfies the equation $2(g(x))^2 - g(x) + x = 0$. Solving this quadratic equation (and throwing out the extraneous solution as Brualdi did at the bottom of page 252), we get $g(x) = \frac{1-\sqrt{1-8x}}{4}$. (And if we want to check that the solution thus far is correct, we can see if $4g(x) = 1 - \sqrt{1-8x}$, i.e. if $(1-4g(x))^2 = 1 - 8x$: so we compute $1 - 4g(x) = 1 - 4(1x + 2x^2 + 8x^3 + 40x^4 + \ldots) = 1 - 4x - 8x^2 - 32x^3 - 160x^4 - \ldots$ and $(1 - 4g(x)) = (1 - 4x - 8x^2 - 32x^3 - 160x^4 - \ldots)^2 = 1 - 8x + 0x^2 + 0x^3 + 0x^4 + \ldots$, as desired.) So $g(x) = \frac{1}{4} - \frac{1}{4}(1-8x)^{1/2}$.

At this point, we could push ahead and imitate the entire rest of the analysis on page 253, and this will give the right answer; I'll give the details below. But if we're lazy and clever (a good combination of personality traits for a mathematician to have!), we'll look for a shortcut, and notice that Brualdi has already done the hard work for us: he's given the power series expansion of $(1-4x)^{1/2}$. So all we have to do is replace x by $2x$:

$$\begin{aligned} (1-8x)^{1/2} &= (1-4(2x))^{1/2} \\ &= 1 - 2\sum_{n=1}^{\infty} \frac{1}{n}\binom{2n-2}{n-1}(2x)^n \end{aligned}$$

$$= 1 - 2\sum_{n=1}^{\infty} \frac{1}{n}\binom{2n-2}{n-1} 2^n x^n.$$

Thus

$$\begin{aligned} g(x) &= \frac{1}{4} - \frac{1}{4}(1-8x)^{1/2} \\ &= \frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{n}\binom{2n-2}{n-1} 2^n x^n \\ &= \sum_{n=1}^{\infty} \frac{1}{n}\binom{2n-2}{n-1} 2^{n-1} x^n. \end{aligned}$$

Hence $h_n = \frac{1}{n}\binom{2n-2}{n-1}2^{n-1}$.

Second solution: Proceed as above to derive the generating function $g(x) = \frac{1}{4} - \frac{1}{4}(1-8x)^{1/2}$, and then apply the binomial theorem:

$$\begin{aligned} (1-8x)^{1/2} &= 1 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n \times 2^{2n-1}}\binom{2n-2}{n-1}(-1)^n 8^n x^n \\ &= 1 + \sum_{n=1}^{\infty} (-1)^{n-1}\frac{2^{n+1}}{n}\binom{2n-2}{n-1} x^n \\ &= 1 - 2\sum_{n=1}^{\infty} \frac{2^n}{n}\binom{2n-2}{n-1} x^n. \end{aligned}$$

Thus

$$g(x) = \frac{1}{4} - \frac{1}{4}(1-8x)^{1/2} = \sum_{n=1}^{\infty} \frac{2^{n-1}}{n}\binom{2n-2}{n-1} x^n$$

and hence $h_n = \frac{2^{n-1}}{n}\binom{2n-2}{n-1}$.

Third solution: If we play around with the way the numbers factor, we might notice that a_2 is divisible by 2, a_3 is divisible by 4, a_4 is divisible by 8, a_3 is divisible by 16, and so on; that is, we might guess that a_n is divisible by 2^{n-1} for all n. If, following this hint, we divide a_n by 2^{n-1}, we get the sequence 1, 1, 2, 5, 14, 42, 132, 429, ..., which we recognize as the sequence from section 7.6. Since the nth term of that sequence is given by the formula $h_n = \frac{1}{n}\binom{2n-2}{n-1}$, we make the guess that $a_n = 2^{n-1}h_n = \frac{1}{n}\binom{2n-2}{n-1}2^{n-1}$.

To prove that $a_n = 2^{n-1}h_n$, we use induction: First, note that the formula is true for $n = 1$, since $a_1 = 1 = 2^0 h_1$. Now, suppose for purposes of induction that the formula is true for all $k < n$. Then we have

$$\begin{aligned} a_n &= 2(a_1 a_{n-1} + a_2 a_{n-2} + \ldots + an - 1a_1) \\ &= 2[(2^0 h_1)(2^{n-2}h_{n-1}) + (2^1 h_2)(2^{n-3}h_{n-2}) + \ldots + (2^{n-2}h_{n-1})(2^0 h_1)] \\ &= 2[2^{n-2}(h_1 h_{n-1} + h_2 h_{n-2} + \ldots + h_{n-1}h_1)] \\ &= 2^{n-1}h_n, \end{aligned}$$

so that the formula is true for n as well. Hence by induction the claim holds for all n.

B. *Chapter 7, problem 22. (Hint: Label the points 1 through $2n$. Let $h_{n,k}$ be the number of ways to join the points in pairs so that the resulting line segments do not intersect, where point 1 is joined to point k. Show that $h_{n,k} = 0$ when k is odd, and find a formula for $h_{n,k}$ in terms of $h_1, h_2, \ldots, h_{n-1}$ when k is even. Use this to write h_n as a sum of products of earlier terms of the sequence.) You may find it convenient to define $h_0 = 1$.*

If point 1 is joined to point k by a chord, then the only way to join up the remaining $2n - 2$ points with $n - 1$ more chords so that no two of the n chords intersect is to pair up the points $2, 3, \ldots, k - 1$ among themselves and to pair up the points $k + 1, k + 2, \ldots, 2n$ among themselves, since any chord joining one of the points $2, 3, \ldots, k - 1$ to one of the points $k + 1, k + 2, \ldots, 2n$ would have to cross the chord we have already drawn between point 1 and point k. If k is odd, so that there are an odd number of points on each side of the chord joining point 1 and point k, then there are no ways to do this: $h_{n,k} = 0$. If k is even, so that there are an even number of points on each side of the chord joining point 1 and point k, then the number of ways to draw those chords is equal to the number of ways to join up the $k - 2$ points $2, 3, \ldots, k-1$ with non-intersecting chords times the number of ways to join up the $2n-k$ points $k+1, k+2, \ldots, 2n$ with non-intersecting chords: $h_{n,k} = h_{(k-2)/2}h_{(2n-k)/2}$. Hence $h_n = h_{n,2} + h_{n,4} + h_{n,6} + \ldots + h_{n,2n} = h_0 h_{n-1} + h_1 h_{n-2} + h_2 h_{n-3} + \ldots + h_{n-1}h_0$.

C. *Chapter 7, problem 41.*

The exponential generating function is $h_0+h_1x+(h_2/2!)x^2+(h_3/3!)x^3+\ldots = 1+\alpha x+\frac{\alpha(\alpha-1)}{2!}x^2+\frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3+\ldots = (1+x)^\alpha$.

D. *Chapter 7, problem 42(c).*

The first factor is $x+x^2/2!+x^3/3!+\ldots$, or e^x-1; the second factor is $x^2/2!+x^3/3!+x^4/4!+\ldots$, or e^x-1-x; the third factor is $x^3/3!+x^4/4!+x^5/5!+\ldots$, or $e^x-1-x-x^2/2!$; etc., up through the kth factor, which is $x^k/k!+x^{k+1}/(k+1)!+\ldots$, or $e^x-1-x-x^2-\ldots-x^{k-1}/(k-1)!$.

E. *Chapter 7, problem 44.*

Solution: The generating function is $(1+x^2/2!+x^4/4!+\ldots)^2(1+x+x^2/2!+x^3/3!+\ldots)^2$, where the first squared factor corresponds to red and green (the colors that must occur an even number of times) and the second squared factor corresponds to blue and orange (the colors that can occur any number of times). This becomes $\left(\frac{e^2+e^{-x}}{2}\right)^2 e^{2x} = \left(\frac{e^{2x}+1}{2}\right)^2 = \frac{1}{4}(e^{4x}+2e^{2x}+1) = \frac{1}{4}(\sum_{n=0}^{\infty} 4^n\frac{x^n}{n!}+2\sum_{n=0}^{\infty} 2^n\frac{x^n}{n!}+1)$. Hence $h_0 = \frac{1}{4}(1+2+1) = 1$ and $h_n = \frac{4^n+2\times 2^n}{4}$ for $n>0$,

Note that the formula $h_n = \frac{4^n+2\times 2^n}{4}$ is not valid for $n=0$.

(If you made the mistake of giving the answer $h_n = \frac{4^n+2\times 2^n+1}{4}$, which is valid *only* for $n=0$, make sure you see where your mistake was. The $+1$ term, viewed as an exponential generating function, should be thought of as $1x^0/0!+0x^1/1!+0x^2/2!+\ldots$, so it contributes only to the value of h_0, not to the values of h_1, h_2, etc.)

F. *Chapter 7, problem 46.*

The exponential generating function for the sequence $h_0, h_1, h_2, \ldots$ is $(1+x^2/2!+x^4/4!+\ldots)^2(x+x^2/2!+x^3/3!+\ldots)^2(1+x+x^2/2!+x^3/3!+\ldots)^2$ where the first squared factor corresponds to 4's and 6's, the second squared factor corresponds to 5's and 7's, and the third squared factor corresponds to 8's and 9's. This becomes $\left(\frac{e^x+e^{-x}}{2}\right)^2(e^x-1)^2 e^{2x} = \left(\frac{e^{2x}+1}{2}\right)^2(e^x-1)^2 = \frac{1}{4}(e^{4x}+2e^{2x}+1)(e^{2x}-2e^x+1) = \frac{1}{4}(e^{6x}-2e^{5x}+3e^{4x}-4e^{3x}+3e^{2x}-2e^x+1)$, so $h_0 = \frac{1}{4}(1-2+3-4+3-2+1) = 0$ and $h_n = \frac{1}{4}(6^n-2\times 5^n+3\times 4^n-4\times 3^n+3\times 2^n-2\times 1^n)$ for $n>0$.