Math 475, Problem Set #1: Solutions

A. *Section 1.8, problem 3. (Hint #1: Use a coloring-argument, as in section 1.1. Hint #2: Try playing with smaller even-by-even rectangles if 8-by-8 seems too big.)*

   The prisoner cannot escape. Imagine coloring the cells alternately black and white, as in a chessboard, so that the square the prisoner starts at is white. He wants to get to the opposite white cell. The total number of moves he makes must be even, since each move takes him from a white cell to a black cell or vice versa, and since his goal and his starting point have the same color. On the other hand, the total number of moves he makes must be 63, since he must visit each of the 64 cells exactly once, and each move takes him to a new cell. Since 63 is not even, we have reached a contradiction. Hence no such escape-route exists.

B. *Section 1.8, problem 4(a).* Every cover (that is, every perfect cover by dominoes) of a 2-by-n rectangle either (A) has a single verticle domino at the right or else (B) a pair of horizontal dominoes at the right. Hence $f(n) = a(n) + b(n)$, where $a(n)$ is the number of covers with a single verticle domino at the right and $b(n)$ is the number of covers with a pair of horizontal dominoes at the right. But $a(n) = f(n-1)$, because there's a one-to-one correspondence between the set of covers of the 2-by-n rectangle of type A and the set of all covers of the 2-by-$(n-1)$ rectangle: in one direction of the correspondence, you just delete the rightmost domino, and in the other direction, you just add a vertical domino at the right. Likewise $b(n) = f(n-2)$. Therefore $f(n) = f(n-1) + f(n-2)$ (for all $n \geq 3$). Starting from $f(1) = 1$ and $f(2) = 2$ and using this *recurrence relation* we successively compute $f(3) = 3$, $f(4) = 5$, $f(5) = 8$, $f(6) = 13$, $f(7) = 21$, $f(8) = 34$, $f(9) = 55$, $f(10) = 89$, $f(11) = 144$, and $f(12) = 233$.

   These are the *Fibonacci numbers*, and they turn up very often in combinatorics and other parts of mathematics. Note that if we posit that $f(2) = f(1) + f(0)$, we get $f(0) = f(2) - f(1) = 2 - 1 = 1$, so that the number of "covers of the 2-by-0 rectangle" is 1.

C. *Repeat the preceding problem, with a twist. Now $f(n)$ counts the number of different perfect covers of a 1-by-n (not 2-by-n!) chessboard by colored tiles of three kinds: a red 1-by-1 tile, a blue 1-by-1 tile, and a green 1-by-2 tile.*

Now we have $f(n) = 2f(n-1)+f(n-2)$ because there are two ways to turn a tiling of size $n-1$ into a tiling of size n, depending on whether the tile we add at the right is red or blue. Starting from $f(1) = 2$ and $f(2) = 5$ and using the recurrence relation we successively compute $f(3) = 12$, $f(4) = 29$, $f(5) = 70$, ..., $f(12) = 33461$.

(Some of you may have noticed a pattern in this sequence: If you divide each term by the preceding term, you get ratio that get closer and closer to 2.41421356..., which looks suspiciously like 1 more than the square root of 2. Later in the course we will see why this is true.)

It's tedious to do these calculations by hand, and even with a calculator you're likely to make a mistake. One way to reduce the risk of numerical error is to use an Excel spreadsheet, and to define cell 1 as 2, cell 2 as 5, and cell 3 as to cell 1 plus twice cell 2; if you copy this definition into cells 4 through 12, you'll see that cell 4 is equal to cell 2 plus twice cell 3, and cell 5 is equal to cell 3 plus twice cell 4, etc.; then you can just read off the answer from cell 12.

Or, if you program in Maple, you can write the recursive program

```
f := proc(n) option remember; if n = 1 then 2;
elif n = 2 then 5; else f(n-2) + 2*f(n-1); fi; end;
```

Typing `f(12)` will give the response `33461`.

D. *Section 1.8, problem 21.*

Whatever color (call it x) is used for region 10 cannot be used for regions 1, 2, 3, 6, 9, 8, 7, and 4. So these eight regions must be colored with the two remaining colors. Say region 1 has color y. Then region 2 must be given the third color, z. Hence three colors are required. Moreover, region 3 is adjacent to regions 2 and 10, so it cannot be colored z or x; if only three colors are permitted, region 3 must have color y. Similarly, in any three-coloring, region 6 must have color z (since it is adjacent to region 10, which is colored x, and region 3, which is colored y). And so on: regions 1, 2, 3, 6, 9, 8, 7, and 4 must have colors y, z, y, z, y,

z, y, and z, respectively. This leaves region 5 adjacent to regions that are all colored z, so there are two choices for the coloring of region 5: either x or y.

Now, x can be red, white, or blue. Once x has been chosen, there are two choices for y. This leaves no freedom in choosing z (it must be the third color), but we have an extra twofold choice in coloring region 5 (we can use either x or y). Hence the total number of colorings is 3 times 2 times 2, or 12.

E. *Section 1.8, problem 26.*

We imitate the solution given in the book to the analogous problem on the 4-by-4 chessboard. Let x_1, x_2, x_3, x_4 and x_5 be, respectively, the number of dominoes which are cut by the five horizontal lines passing in the interior of the 6-by-6 chessboard. Suppose there is no fault line. Then each of x_1 through x_5 is positive. Also, each of them must even. (Proof: The area above each horizontal line is even, and the combined area of the dominoes above the line that aren't cut by the line is even, so the difference between the two must be even. But this difference is equal to the number of dominoes that are cut by the line.) Hence $x_1 + x_2 + x_3 + x_4 + x_5 \geq 2 + 2 + 2 + 2 + 2 = 10$, so there are at least 10 vertical dominoes, with total area at least 20. Likewise there are at least 10 horizontal dominoes, with total area at least 20. But the total area of the square is 36, which is less than $20 + 20$: contradiction.

F. *Twenty-five students are seated in a square arrangement with 5 rows of 5 desks each. The teacher tells all the students to switch desks so that every student is switched to a desk either directly to the front, to the back, to the right, or to the left of the original desk. Can all the students switch to a new desk simultaneously?*

Color the seats black and white checkerboard-fashion. When the teacher has the students switch desks, every students at a white desk goes to a black desk, and vice versa. However, there are unequal numbers of black and white desks, so it is impossible for all the students to switch.

Alternative solution #1 (Ari Trachtenberg, MIT): Number the desks by rows from 1 to 25. When a student switches desks, his desk-number changes by an odd number (± 1 or ± 5). Since there are an odd number of students, and since the sum of an odd number of odd numbers must be odd, the sum of the students' desk-numbers must change by an odd number. But notice that the sum of the students' desk-numbers is just the sum $1 + 2 + ... + 25$, both before and after; that is, it has changed by 0 (which is not an odd number).

Alternative solution #2 (David Pecora, MIT): The students may be divided up into cycles, with the property that each student moves to the desk of the next student in the same cycle. Since there are 25 students, at least one of the cycles must have odd length. But observe that as one follows a cycle, every northward step must be balanced by a southward step, and every eastward step must be balanced by a westward step. Hence the total number of steps in each cycle must be even. Contradiction.

Alternative solution #3: Imagine the first (starting) cell and the second cell visited by the prisoner as forming a domino, and likewise for the third and fourth cells, fifth and sixth cells, etc. Since the prisoner must visit each cell exactly once, this leads to a perfect cover of the 8-by-8 square by dominoes. But now suppose that, as he goes, he paints each cell he visits, alternately using the colors black and white, starting with black. Since each time he moves to a new cell he moves to an adjoining cell and changes colors, the coloring he creates must be the standard coloring of the chessboard, with a black cell in the upper left. But notice that if this could be done, it would lead to a perfect cover of the either the 62-square region formed by removing both corners or the 63-square region formed by removing just the upper corner.