

Math 192r, Problem Set #10
(due 10/23/01)

1. Using a sign-reversing involution, prove that for all $n > k$, the sum $\sum_{m: k \leq m \leq n} s(n, m)S(m, k)$ equals zero.
2. Consider the subset of the square grid bounded by the vertices $(0, 0)$, $(m, 0)$, $(0, n)$, and (m, n) , and let q be a formal indeterminate. Let the weight of the horizontal grid-edge joining (i, j) and $(i + 1, j)$ be q^j (for all $0 \leq i \leq m - 1$ and $0 \leq j \leq n$), and let each vertical grid-edge have weight 1. Define the weight of a lattice path of length $m + n$ from $(0, 0)$ to (m, n) to be the product of the weights of all its constituent edges. Let $P(m, n)$ be the sum of the weights of all the lattice paths of length $m + n$ from $(0, 0)$ to (m, n) , a polynomial in q . (Note that putting $q = 1$ turns $P(m, n)$ into the number of lattice paths of length $m + n$ from $(0, 0)$ to (m, n) , which is the binomial coefficient $\frac{(m+n)!}{m!n!}$.)

- (a) Give a formula for $P(1, n)$ and for the generating function

$$\sum_{n \geq 0} P(1, n)x^n.$$

- (b) Find (and justify) a recurrence relation relating the polynomials $P(m, n)$, $P(m - 1, n)$, and $P(m, n - 1)$ that generalizes the Pascal triangle relation for binomial coefficients.
- (c) Let $F_m(x)$ denote $\sum_{n \geq 0} P(m, n)x^n$. Use your answer from (b) to give a formula for $F_m(x)$ in terms of $F_{m-1}(x)$, and from this derive a non-recursive formula for $F_m(x)$.
- (d) Write a computer program to compute the polynomial $P(m, n)$ for any input values m, n .
- (e) Compute $P(m, n)/P(m - 1, n)$ for various values of $m \geq 1$ and $n \geq 0$ and conjecture a formula for it. Do the same for the ratio $P(m, n)/P(m, n - 1)$ with $m \geq 0$ and $n \geq 1$.
- (f) Use the recurrence relation from (b) to verify your conjectures from (e).